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of
Nonsmooth First Order Hyperbolic Systems

Theory and Application

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Abstract

In this thesis problems of optimal control subject to partial differential equations and variational inequalities of stationary and evolutionary type are studied where the differential operator is of first order.

We introduce a model for open pit mine planning based on continuous functions describing profiles and reformulate the problem in such a way, that the pointwise constraint ensuring physical stability of the profiles is satisfied by viscosity solutions of a certain nonlinear first order partial differential equation of Eikonal type.

We study the resulting problem of optimal control subject to this equation and establish the existence of solutions. Utilizing the vanishing viscosity approach we in addition consider auxiliary optimizations problems subject to semilinear partial differential equations and use the increased regularity of their solutions to establish, besides the theory concerning the existence of solutions, a first order necessary optimality condition. Finally we present a mild consistency result of solutions to the auxiliary problems.

Further we study problems of optimal control subject to stationary variational inequalities of the first kind with linear first order differential operators. We discuss known existence results of the underlying problems with respect to the optimal control setting and present results concerning solvability for regularized problems. The regularization does not influence the nature of the underlying problem which remains of first order. We achieve this by extending a result concerning solutions to variational inequalities with degenerate differential operators. Utilizing a vanishing viscosity approach we use established penalization and regularization techniques for the resulting problems of optimal control subject to elliptic variational inequalities and establish a certain kind of stationarity system for the original problems which is weaker than W stationarity. It is derived under certain boundedness assumptions when considering the limit of the viscosity parameter. We discuss the results by numerical studies for several examples.

In the last part we extend the results from the stationary setting to linear first order problems of evolutionary type. Again solvability of the problems is proven for regularized problems. We show that the solutions of the underlying first order variational inequalities can be approximated by solutions to parabolic ones Again we utilize a vanishing viscosity approach and derive a stationarity system for the first order problems. Subsequently the result is obtained using again certain boundedness assumptions.

In both of the latter cases we concentrate on the cone conditions arising from obstacle problems.

Abstract

In dieser Arbeit betrachten wir Optimalsteuerungsprobleme von Lösungen für Differentialgleichungen und Variationsungleichungen erster Art sowohl im stationären als auch im zeitabhängigen Fall. Hierbei werden die genannten Objekte durch einen Differentialoperator erster Ordnung charakterisiert.

Wir erweitern ein Modell für optimale Tagebauplanung, das auf stetigen Funktionen beruht welche die Profile der Mine beschreiben. Dabei wird das Problem in einer Art und Weise reformuliert, dass die punktweise Bedingung an die Profile, welche die physikalische stabilität des Tagebaus sichert, dann erfüllt ist, wenn die Profile aus Viskositätslösungen einer eikonalartigen nichtlinearen Differentialgleichung erster Ordnung rekonstruiert werden. Wir diskutieren das resultierende Optimalsteuerungsproblem und weisen die Existenz von Lösungen nach. Unter Verwendung der Methode der verschwindenden Viskosität definieren wir des Weiteren Hilfsprobleme, deren unterliegende Gleichung semilinear ist. Wir benutzen die dadurch gewonnene erhöhte Regularität der Zustände um neben dem Nachweis der Existenz von Lösungen eine notwendige Optimalitätsbedingung erster Ordnung herzuleiten. Schlussendlich präsentieren wir ein schwaches Konsistenzresultat für die regularisierten Probleme.

Des Weiteren diskutieren wir Optimalsteuerungsprobleme für lineare, stationäre Variationsungleichung erster Art mit einem Differentialoperator erster Ordnung. Wir betrachten bekannte Resultate über die Existenz von Lösungen solcher Ungleichungen unter dem Gesichtspunkt der optimalen Kontrolle und präsentieren ein Resultat, das die Existenz von Lösungen des Steuerungsproblems sichert. Hierbei muss zwar ein zusätzlicher Term in das Zielfunktional aufgenommen werden, aber die Natur des Differentialoperators wird nicht verändert. Dafür erweitern wir bekannte Resultate für Variationsungleichungen mit degenerierten Differentialoperatoren. Darüber hinaus nutzen wir die Methode der verschwindenden Viskosität um ein Optimalsteuerungsproblem für elliptische Variationsungleichungen zu erhalten, das mit bekannter Theorie auf die Charakterisierung von stationären Punkten hin untersucht wird. Mit Hilfe dieser Charakterisierung erhalten wir eine gewisse Art von Stationaritätssystem für das Ausgangsproblem, das schwächer als im elliptische Fall ist und nur unter der Annahme einer bestimmten Beschränktheit für stationäre Punkte der Hilfsprobleme gewonnen werden kann. Wir untermauern die Überlegungen mit der numerischen Betrachtung von Beispielen.

Im letzten Teil der Dissertation erweitern wir die Ergebnisse für die stationären Probleme auf die Optimalsteuerung von zeitabhängigen Variationsungleichungen erster Art mit linearem Differentialoperator erster Ordnung. Die Lösbarkeit der Probleme wird erneut für regularisierte Probleme nachgewiesen, wobei die Ordnung der zugrundeliegenden Variationsungleichung erhalten bleibt. Wir weisen nach, dass die Lösungen der Ungleichungen erster Ordnung in diesem Fall durch Lösungen parabolischer Variationsungleichungen mit der Methode der verschwindenden Viskosität approximiert werden können indem wir erneut ein Ergebnis über degenerierte Differentialoperatoren erweitern. Diesen Zusammenhang nutzen wir aus und leiten aus den zugehörigen Stationaritätssysthemen für die Hilfsprobleme ein schwaches Stationaritätssystem für das Ausgangsproblem her, wobei wieder eine bestimmte Beschränktheit angenommen werden muss.

Contents

1 Introduction

The main focus of this thesis lies on problems of optimal control where the underlying constraint involves a first order differential operator either depending on an additional parameter, usually referred to as time, or not. We will investigate such problems in three different settings.
The first first example we discuss is taken from open pit mine planning. In this area, the common practice of formulating optimization problems is based on establishing a three dimensional block model of the deposit. Such block models are constructed by partitioning the given volume into a huge number of small blocks each of which are assigned a price and an effort. Here the price represents the net gain when the material within the block is processed or dumped and therefore can have negative values while the strictly positive effort quantifies the investment of time, equipment and energy that has to be made to excavate the material in this block. Information about these quantities are usually obtained by geostatistical methods as in [50] and assumed to be time independent. In any case real data at certain coordinates in the volume are used to obtain the distribution of gain and effort in the whole deposit. The real data stem from cuts and test drills at the considered coordinates. We point out, that the size of elementary blocks has to be considered in relation to the dimension of the whole mine. The models with highest resolution have blocks of 25 meters along each edge but the mines are up to 4.3 km by 3 km on the surface and up to 1 km deep (see for example Chuquicamata (Chile)). As a consequence, the number of blocks is enormous. The resulting model can be interpreted as a directed graph where the nodes represent the blocks of the model endowed with weights representing gain and effort. The edges represent certain dependencies of the blocks and determine, if or when a block can be excavated. Here the most natural condition is, that a block can only be excavated when all blocks above it have already been removed. In addition, one usually requires, that certain blocks in the horizontal layer above the current one have to be already excavated to avoid steep slopes in the profiles and thus to ensure the physical stability of the mine. This introduces the so called cone of dependencies for each node in the graph. The open pit mine planning problem in graph form is to find a sequence of blocks to be excavated such that the total gain, time dependent or not, is maximized while the total effort, the mining operation is able invest, is not exceeded. This is an NP hard problem as shown in [42].
The discrete problems have been tackled by a wide variety of methods. Natural approaches are of combinatorial nature and integer or mixed integer programming techniques as in [28]. A huge number of publications considering the discrete model have appeared since 1960 starting with the pioneering paper [101]. Here an algorithm of polynomial run time was presented which computes the so called ultimate (gain) pit defined as the profile whose excavation yields the maximal revenue. Although disregarding the temporal evolution, the algorithm has been used for a very long time in industry. Problem instances also incorporating time by considering a discount function for the gain and subject to the effort constraint in each time step are much more difficult to handle. In [21, 29, 84] they have been studied with techniques from discrete optimization. Moreover, they were considered as dynamical Programs in [88, 155] and tackled by metaheuristics and evolutionary algorithms in [44, 54].

A different perspective on the problem comes from considering the profiles of the pit as graph of a function. Here the mechanical stability of the mine is translated into a local condition on the function. To our knowledge, the first works considering such a model were [114] and [113] where the stability condition was represented by a cone which has to lie above the graph of the current profile. However, this approach has not been investigated further and this point of view was not considered for almost 40 years. In [4] and [62] it was revived and optimal shapes of the mine were searched among Lipschitz continuous functions. In this setting the stability condition translates into a bound on the Euclidean norm of the gradient of the profile which has to be satisfied almost everywhere. Recently another approach has been published in [47] where the overall problem was formulated as a problem of optimal transportation.
In Chapter 5 we will further develop the model originally presented in [4]. The new approach reformulates the continuous model for the time depending case and provides an optimization problem of optimal control subject to a first order hyperbolic partial differential equation of Eikonal type. This part of the thesis will mostly work in the context of classical theory for partial differential equations.
It is known, that general hyperbolic problems do not admit smooth solutions in general (see [108]). Therefore generalized solution concepts have been considered. Entropy solutions were introduced in [96] for a certain class of problems covering in particular scalar conservation laws. They are weak solutions in the sense of distribution. A further concept are continuous viscosity solutions as discussed in [108]. They play an important role in the context of optimal control of dynamical systems since the value functional of such problems is given as the viscosity solution to a certain, problem dependent first order equation also referred to as Hamilton-Jacobi equation (see for example [9]).
Theoretical results concerning optimal control first order partial differential equations are rather scarce. In [43] the Eikonal equation was controlled in the framework of etching out a certain surface.
In the last years, especially the optimal control of conservation laws has been investigated. There are several references dealing with this kind of problems as for example in [36, 151].
This thesis will focus on viscosity solutions but Chapter 4 contains an overview on generalized solution concepts, their relation and known applications. Both generalized concepts are obtained by regularizing the differential, operator with an artificial, weighted viscosity term. Then the convergence of solutions of the regularized equations with the weight tending to zero is studied in certain topologies and the two mentioned concepts are obtained. Concerning the optimal control of viscosity solutions we refer to the very recent work [60]. Neither the optimal control of conservation laws nor the last given reference provided useful tools for the study of our problem. We will prove that if the first order equation in the model of open pit mine planning admits a viscosity solution, the profiles constructed from this function are physically stable. We will investigate the corresponding problems of optimal control both in the regularized and non regularized form and proof the existence of solutions. Finally we will derive a first order optimality condition for regularized problems.
A second area of interest for problems with a first order differential operator are variational inequalities. They were introduced in [14] in the context of deterministic control problems. Concerning viscosity solutions of such problems it was shown in [13], that a solution exist if it can be described as upper value of a certain differential game with stopping time. They are of great interest in the context of optimal control of dynamical systems if an additional state constraint is introduced to the problem for the following reason. As mentioned before, the optimal value functional of the problem is the viscosity solution of a Hamilton-Jacobi equation.

Under additional state constraints for the dynamical system, the underlying problem turns into a variational inequality. Besides a closed and convex constraint for the state of the system at all times as in [14], variational inequalities also occur if one introduces a feasible target region as in [20].

In the classical setting, variational inequalities with first order differential operators have also been investigated for the generalized concept of entropy solutions. We refer to [103, 104] and the references therein for further information about this case. Besides the regular solutions, weak solution concepts were studied as well. They were developed in [118] and further investigated for example in [133, 135]. Here problems of the form

$$\text{find } y \in \mathbf{K} : \langle \mathbf{b} \cdot \nabla y + b^0 y - f, v - y \rangle \geq 0 \text{ for all } v \in \mathbf{K}.$$

are formulated in the Hilbert space $L^2_B(\Omega)$ specified by the problem. For the definition of $\mathbf{K}$ we refer to Chapter 4. Unfortunately, this concept is to weak in a certain sense for application in optimal control as we will discuss in Chapter 6. There we usually have to fit some data that are only elements of $L^2(\Omega)$. To ensure the existence of solutions to the overall optimization problem

$$\begin{array}{ll} \inf & \frac{1}{2}|y - y^d|^2_{L^2(\Omega)} + \frac{\beta}{2}|u|^2_{L^2(\Omega)} \\ \text{s.t.} & y \in \mathbf{K}, \;\; \langle \mathbf{b} \cdot \nabla y + b^0 y - f - u, v - y \rangle \geq 0 \quad \forall v \in \mathbf{K} \end{array}$$

we need the solutions of the underlying object to be elements of $H^1_0(\Omega)$. In Chapter 6 we will transfer the concept of viscosity solutions to the setting of weak solutions of stationary variational inequalities of the first kind with first order differential operators. We will utilize the observations of the preceding Chapter and introduce a further Tikhonov regularization of the state for the space $H^1_0(\Omega)$ and prove the existence of solutions under certain conditions. Moreover we will establish a weak kind of stationarity system for the problem. This will be done in two different Hilbert space settings for the states, namely $L^2_B(\Omega)$ and $H^1_0(\Omega)$, where the latter one is strongly related to the study of auxiliary problems defined as

$$\begin{array}{ll} \min & \frac{1}{2}|y - y^d|^2_{L^2(\Omega)} + \frac{\beta}{2}|u|^2_{L^2(\Omega)} \\ \text{s.t.} & y \in \mathbf{K}, \;\; \langle -\varepsilon \Delta y + \mathbf{b} \cdot \nabla y + b^0 y - f - u, v - y \rangle \geq 0 \quad \forall v \in \mathbf{K} \end{array}$$

Stationarity concepts of the auxiliary optimization problems subject to elliptic variational inequalities in function spaces were intensively studied in the last decades. They are mostly based on the finite dimensional counterparts which are introduced and broadly discussed in [53, 111]. These finite dimensional concepts have been transferred to a Hilbert space setting in [73, 95] where a hierarchy of them was established as well. Moreover, they were embedded into already established theory as presented in [117, 119]. Further information about stationarity concepts are collected in Chapter 3. Concerning optimal control of elliptic variational inequalities we refer to [7, 8, 73, 119] and the references therein.

In the last part we will extend the results form Chapter 6 to the time dependent case where the differential operator of the variational inequality also depends on a parameter usually interpreted as time. There are several publications on the control of hyperbolic variational inequalities (see for example [126, 148]) but all of them are related to problems of second order in time. In the case of first order operators the underlying problem is given as

$$\text{find } y \in \mathcal{K} : \langle D_t y + \mathbf{b} \cdot \nabla y + b^0 y - f - u, v - y \rangle \geq 0 \;\; \forall v \in \mathcal{K}.$$

It is more challenging than the problems from Chapter 6 since coercivity of the differential operator can not be established in general. The concept of weak solutions in the space $L^2_B(\Omega)$ has been extended in [134] to the time dependent case but the drawbacks concerning problems of optimal control remain in this setting as well. The function spaces for the consideration of such variational inequalities will be introduced in Chapter 2 and further discussed in Chapter 7. We will prove the existence of solutions for such objects with constraint sets of the type

$$\mathcal{K} = \{v \in \mathfrak{V} | v(t) \geq \psi \text{ for a.e. } t \in (0,T)\} = \{v \in \mathfrak{V} | v(t) \in \mathbf{K}\}$$

where $\mathbf{K} \in H^1_0(\Omega)$ does not vary in time. Parabolic variational inequalities with symmetric differential operators and constraint sets varying over time have for example been considered in [120]. The problems of optimal control we are going to investigate are given as

$$\begin{array}{lrll} \inf & \frac{1}{2}|y - y^d|^2_{L^2(\mathcal{Q})} + \frac{\beta}{2}|u|^2_{L^2(\mathcal{Q})} & & \\ \text{s.t.} & \langle D_t y + \mathbf{b} \cdot \nabla y + b^0 y - f - u, v - y \rangle & \geq 0 & \forall v \in \mathcal{K} \\ & y(0) & = u_0 & \in H^1_0(\Omega) \cap \mathbf{K} \end{array}$$

Again we have to introduce a Tikhonov regularization for the states to ensure solvability of the problem but in this case it is sufficient to do this in the space $L^2(0,T;H^1_0(\Omega))$ instead of the subspace where solutions to second order evolutionary problems usually live. As in Chapter 6 we will regularize the differential operator and consider the following family of auxiliary problems.

$$\begin{array}{lrll} \min & \frac{1}{2}|y - y^d|^2_{L^2(\mathcal{Q})} + \frac{\tilde{\beta}}{2}|y|^2_{L^2(0,T;H^1_0(\Omega))} + \frac{\beta}{2}|u|^2_{L^2(\mathcal{Q})} & & \\ \text{s.t.} & \langle D_t y - \varepsilon \Delta y + \mathbf{b} \cdot \nabla y + b^0 y - f - u, v - y \rangle & \geq 0 & \forall v \in \mathcal{K} \\ & y(0) & = u_0 & \in H^1_0(\Omega) \cap \mathbf{K} \end{array}$$

For the stationarity results of this problems we will also rely on existing theory from [95]. While the existence theory of solutions for the underlying objects is derived for several choices of $\mathbf{K}$ we will focus on inequalities of obstacle type only in the optimal control part. We will derive a certain kind of stationarity system which is weaker than the one obtained in the stationary case from Chapter 6 due to the lack of coercivity of the differential operator.
The results established in this thesis concerning open pit mine planning have been presented at international conferences and parts were published in [63].

2 Preliminaries

In this chapter we introduce the basic framework for partial differential equations and variational inequalities and define the function spaces of interest for this thesis. They include, apart from Lebesgue and Bochner spaces, the spaces of Hölder continuous functions. All involved spaces are defined for an open, bounded domain $\Omega \subset \mathbb{R}^n$ with $1 \leq n \leq 3$. Whenever we make further restrictions on the domain this will be pointed out explicitly.
Section 2.2 introduces partial differential equations in the stationary and evolutionary case for classical and weak solutions. Ordered by problem complexity, they are followed by variational inequalities in Section 2.3.
Finally we introduce approximating techniques for variational inequalities in Section 2.4.

2.1 Function Spaces

In the most scenarios occurring in this thesis we deal with Banach spaces, i.e. normed vector spaces that are complete (every Cauchy sequence converges).
As important exception we use the space of infitely often differentiable functions with compact support $C_c^\infty(\Omega)$ which is complete but not a normed vector space (see [45]).
We consider two basic types of function spaces. On the one hand functions only depending on the domain Ω, suited to problems of stationary type. On the other hand, we use functions additionally depending on a parameter usually interpreted as time. These spaces are the basis for the investigation of evolutionary problems.

2.1.1 Spaces of Continuous Functions for Stationary Problems

Consider

$$(C(\bar{\Omega}), \|\cdot\|_\infty) \quad \text{and } (C^k(\bar{\Omega}), \|\cdot\|_{C^k(\bar{\Omega})})$$

of bounded and continuous real valued functions defined on Ω with the supremum norm $\|\phi\|_\infty = \sup\limits_{x\in\Omega} |\phi(x)|$ and bounded continuous real valued functions with bounded continuous derivatives up to order k respectively. In the latter case, the norm is defined as

$$\|\varphi\|_{C^k(\bar{\Omega})} = \sum_{0 \leq |\pi| \leq k} \|D^\pi \varphi\|_\infty$$

with the multi index $\pi = (\pi_1, ..., \pi_n)$, $|\pi| = \sum_{i=1}^n \pi_i$ and the differential $D^\pi \varphi(x) = \frac{d^\pi \varphi}{dx_1^{\pi_1} ... dx_n^{\pi_n}}$.
Both spaces are Banach spaces (see [57]) and $\overline{\Omega}$ indicates, that function and derivatives have a continuous extension to $\partial\Omega$.
A function φ is called Hölder continuous in Ω with exponent $0 < \theta \leq 1$ if the so called Hölder constant

$$\langle \varphi \rangle^\theta = \sup_{x,y \in \Omega, x \neq y} \frac{|\varphi(x) - \varphi(y)|}{|x - y|^\theta}$$

is finite. With this seminorm we introduce for $0 < \theta \leq 1$ and integer k the function space

$$(C^{k+\theta}(\bar{\Omega}), \|\cdot\|_{C^{k+\theta}(\bar{\Omega})})$$

with the norm

$$\|\varphi\|_{C^{k+\theta}(\bar{\Omega})} = \|\varphi\|_{C^k(\bar{\Omega})} + \sum_{|\pi|=k} \langle D^\pi \varphi \rangle^\theta.$$

The resulting space is a Banach space (see [57, 98]) and the embedding $C^{k+\theta_1}(\bar{\Omega}) \to C^{k+\theta_2}(\bar{\Omega})$ with $\theta_1 \geq \theta_2$ is continuous for all bounded domains Ω. In the case $\theta = 1$, the functions are called Lipschitz continuous having additional properties (see Section A.1).
Hölder continuity is preserved under several univariate and bivariate operations as the next Lemma briefly notes for selected examples.

Lemma 2.1.1. *The following calculus rules for Hölder continuous functions hold*

1. *Let f_1 be Hölder continuous with exponent θ_1 and constant $\langle f_1 \rangle^{\theta_1}$ and f_2 be Hölder continuous with exponent $\theta_2 \geq \theta_1$ and constant $\langle f_2 \rangle^{\theta_2}$. Then $f_1 + f_2$ is Hölder continuous with exponent $\theta = \min\{\theta_1, \theta_2\} = \theta_1$ and*

$$\langle f_1 + f_2 \rangle^\theta \leq \langle f_1 \rangle^{\theta_1} + \langle f_2 \rangle^{\theta_2} \operatorname{diam}(\Omega)^{\theta_2 - \theta_1}$$

2. *Let f_1 be Hölder continuous with exponent θ_1 and constant $\langle f_1 \rangle^{\theta_1}$ and f_2 be Hölder continuous with exponent $\theta_2 \geq \theta_1$ and Corresponding constant $\langle f_2 \rangle^{\theta_2}$. Then $f_1 f_2$ is Hölder continuous with exponent $\theta = \min\{\theta_1, \theta_2\} = \theta_1$ and*

$$\langle f_1 f_2 \rangle^\theta \leq \|f_2\|_\infty \langle f_1 \rangle^{\theta_1} + \|f_1\|_\infty \operatorname{diam}(\Omega)^{\theta_2 - \theta_1} \langle f_2 \rangle^{\theta_2}$$

3. *Let f be a Hölder continuous function with exponent θ and constant c satisfying $f \geq \nu > 0$. Then the reciprocal f^{-1} is Hölder continuous with the same exponent θ and*

$$\langle 1/f \rangle^\theta \leq M/\nu^2$$

4. *Let $f \geq 0$ be a Hölder continuous function with exponent θ and constant c. Moreover, let $\nu > 0$ be given. Then $\sqrt{f + \nu^2}$ is Hölder continuous with exponent $\theta = \theta$ and*

$$\langle \sqrt{f + \nu^2} \rangle^\theta \leq c/(2\nu)$$

5. *Let f_1 be Hölder continuous with exponent θ_1 and constant $\langle f_1 \rangle^{\theta_1}$ and f_2 be Hölder continuous with exponent $\theta_2 \geq \theta_1$ and Corresponding constant $\langle f_2 \rangle^{\theta_2}$. Then for all $\lambda \in [0, 1]$ and $\bar{\lambda} = 1 - \lambda$ the convex combination $\lambda f_1 + \bar{\lambda} f_2$ is Hölder continuous with exponent $\theta \leq \min\{\theta_1, \theta_2\} = \theta_1$*

$$\langle \lambda f_1 + \bar{\lambda} f_2 \rangle^\theta \leq \lambda \langle f_1 \rangle^{\theta_1} + \bar{\lambda} \langle f_2 \rangle^{\theta_2} \operatorname{diam}(\Omega)^{\theta_2 - \theta_1}$$

6. *Let $f(s)$, $s \in [0, 1]$ be a parametrized family of Hölder continuous functions with Corresponding exponents $\theta(s)$ and constants $c(s)$. Let in addition $\theta(s)$ and $c(s)$ be bounded in $[0, 1]$ and $\theta(s) \geq \nu > 0$ for all $s \in [0, 1]$. Then $\int_0^1 f(s) ds$ is Hölder continuous*

with exponent $\theta = \nu$ and

$$\langle \int_0^1 f(s)ds \rangle^\theta \leq \int_0^1 c(s)\,\mathrm{diam}\,(\Omega)^{\theta(s)-\theta}$$

Proof. Positions 1) to 3) follow from basic calculus.
4) follows from the more general statement, that given any Lipschitz continuous function ϕ with constant L we obtain

$$|\phi(f(x_1)) - \phi(f(x_2))|/|x_1 - x_2|^\theta \leq L_\phi |f(x_1) - f(x_2)|/|x_1 - x_2|^\theta \leq L_\phi c$$

and that the Lipschitz constant of the given function is $1/2\nu$.
5) and 6) are standard again. □

Any of the above function spaces can be equivalently defined for functions vanishing on the boundary $\partial\Omega$. In this case the subscript 0 is added to the notation of the function space. Finally we introduce the space of test functions

$$C_c^\infty(\Omega) := \{\varphi \in C^\infty(\Omega) : supp(\varphi) \subset \Omega\}$$

consisting of all smooth functions whose support is compactly contained in the domain Ω. As already mentioned, this space is not a Banach space since it can not be endowed with a norm.

2.1.2 Spaces of Continuous Functions for Evolutionary Problems

For problems of evolutionary type one has to consider, besides the spatial domain Ω, a parametrization describing the evolution of the solution to some differential equation. This parametrization usually is interpreted as time while we will also consider another interpretation in Chapter 5. For evolutionary problems one considers the extended domain

$$\mathcal{Q} = \Omega \times I = \{(x, z) \in \mathbb{R}^{n+1} | x \in \Omega \text{ and } z \in I\}$$

where $I \in \mathbb{R}^1$ is a closed interval representing the domain of the parameter. Here we only use I as a closed, connected and bounded interval $[0, \overline{z}]$ with $\overline{z} > 0$. When the parameter represents time, $\mathcal{Q}$ is called space-time cylinder.
For the introduction of boundary conditions we define the so called parabolic boundary

$$\Gamma = \Omega \times \{0\} \cup \partial\Omega \times I$$

and the set $S = \partial\Omega \times \{0\}$. The space of (Hölder) continuous functions for evolutionary problems is denoted by

$$C^{k+\theta, k/2+\theta/2}(\mathcal{Q}).$$

In this work we will exclusively use $k = 2$ and thus restrict the following definitions to this value. For the general case with arbitrary integer k we refer to [97, 98]. The function space contains all bounded and continuous functions φ on $\mathcal{Q}$ that have bounded and continuous derivatives $D_z^r D_x^\pi \varphi$ for all $2r + |\pi| \leq 2$ on $\bar{\mathcal{Q}}$ and finite values for

$$\langle \varphi \rangle_{x,\mathcal{Q}}^{[2+\theta]} = \sum_{2r+|\pi|=2} \langle D_z^r D_x^\pi \varphi \rangle_x^\theta \quad \text{and} \quad \langle \varphi \rangle_{z,\mathcal{Q}}^{[1+\theta/2]} = \sum_{0 \leq 2r+|\pi| \leq 2} \langle D_z^r D_x^\pi \varphi \rangle_z^{\theta/2}$$

with

$$\langle\tilde{\varphi}\rangle_x^\theta = \sup_{(x,z),(\tilde{x},z)\in\mathcal{Q}} \frac{|\tilde{\varphi}(x,z) - \tilde{\varphi}(\tilde{x},z)|}{|x-\tilde{x}|^\theta} \quad \text{and} \quad \langle\tilde{\varphi}\rangle_z^{\theta/2} = \sup_{(x,z),(x,\tilde{z})\in\mathcal{Q}} \frac{|\tilde{\varphi}(x,z) - \tilde{\varphi}(x,\tilde{z})|}{|z-\tilde{z}|^{\theta/2}}.$$

With

$$\|\varphi\|_{C^{k+\theta,k/2+\theta/2}(\mathcal{Q})} = \sum_{2r+|\pi|\le 2} |D_i^r D_x^\pi \varphi|_\infty + \langle\varphi\rangle_{x,\mathcal{Q}}^{[2+\theta]} + \langle\varphi\rangle_{z,\mathcal{Q}}^{[1+\theta/2]}$$

$C^{k+\theta,k/2+\theta/2}(\mathcal{Q})$ is a Banach space (see [97, 98]). The first index in the exponent defines the regularity of the functions with respect to the spatial variables while the second does the same for with respect to the parameter. Note that $C^{\theta,\theta_2}(\mathcal{Q})$ is well defined for arbitrary combinations $(\theta,\theta_2)\in\mathbb{R}^2$. The presented form $\theta_2=\theta/2$ is suited to parabolic problems. One easily checks the embedding

$$C^\theta(\mathcal{Q}) := C^{\theta,\theta}(\mathcal{Q}) \to C^{\theta,\theta_2}(\mathcal{Q})$$

for $\theta_2 \le \theta$ to be continuous whenever $|\mathcal{Q}|<\infty$. We point out, that this definition differs from the setting

$$C^{k+\theta}(I;X)$$

of bounded continuous functions φ that are k times continuously differentiable selections of elements in a given Banach space X. Here, the quantity

$$\langle\varphi^{(k)}\rangle_{C^\theta(I;X)} = \sup_{z_1,z_2\in I, z_2<z_1} \frac{\|\varphi^{(k)}(z_1)-\varphi^{(k)}(z_2)\|_X}{|z_1-z_2|^\theta}$$

has to be finite and the corresponding norm is then given as

$$\|\varphi\|_{C^{k+\theta}(I;X)} = \sum_{l=0}^k \sup_{i\in I} \|\varphi^{(k)}(i)\|_X + \langle\varphi^{(k)}\rangle_{C^\theta(I;X)}.$$

The following Proposition establishes additional Hölder regularity of a function with respect to time provided it enjoys a certain regularity concerning the spatial variables.

Proposition 2.1.1 ([98]). *Let $\varphi(x,z)$ satisfy a Hölder condition in z with exponent θ_1 and constant c_1 and let it have derivatives φ_x, which for any z from $[0,\overline{z}]$, are Hölder continuous in x with exponent θ_2 and constant c_2. If Ω satisfies the cone property, the derivatives φ_x satisfy in $\mathcal{Q}$ a Hölder condition with exponent $\theta=\theta_1\theta_2/(1+\theta_2)$ and a constant only depending on $\theta_1,\theta_2,c_1,c_2,n$ and the angle of the vertex of the cone.*

The cone condition is the same as in [2], i.e. there exists a finite cone $\mathbf{C}$ such that each $x\in\Omega$ is the vertex of a cone congruent to $\mathbf{C}$ and contained in Ω. The following result provides a criteria for compactness in the space of bounded and continuous functions.

Theorem 2.1.1 (Arzela Ascoli; see [154]). *Let $\Omega\subset\mathbb{R}^n$ with $n\ge 1$ be a bounded domain and $\Theta\subset C(\Omega)$ be a subset of continuous functions that is closed, bounded and equicontinuous, i.e.*

$$\forall\varepsilon>0\,\exists\delta>0 \;\; s.t. \;\forall\varphi\in\Theta \text{ and } x,y\in\Omega : |x-y|\le\delta \Rightarrow |\varphi(x)-\varphi(y)|\le\varepsilon.$$

Then Θ is compact.

As a direct Corollary we find.

Corollary 2.1.1. *Let $\Theta \subset C(\overline{\Omega})$ be a family of Lipschitz continuous functions with a common Lipschitz constant L. Moreover, suppose there exist at least one point $x \in \bar{\Omega}$ with $\varphi(x) = c$ for all $\varphi \in \Theta$.*
Then the family is compact.

Proof. Let L be the upper bound on the Lipschitz constants. Based on

$$|\varphi(x) - \varphi(y)| \leq L|x - y| \leq L\delta \leq \varepsilon$$

we find $\delta > 0$ for any given $\varepsilon > 0$. Thus Θ is equicontinuous. By Lemma A.1.1 we obtain closedness of the set under uniform convergence, i.e. in the norm $\|\cdot\|_\infty$. Finally the common value at a given point x and the common upper bound on the Lipschitz constant provide boundedness of the family Θ. Now the assertion follows from Theorem 2.1.1. □

2.1.3 Lebesgue Spaces for Nonevolutionary Problems

We recall the spaces of Lebesgue measurable functions for an open domain $\Omega \subset \mathbb{R}^n$. For a detailed discussion we refer to [2, 26, 154]. Given $p \in [1, \infty]$ we define the Banach spaces

$$L^p(\Omega) = \{\varphi : \Omega \to \mathbb{R}; \|\varphi\|_{L^p(\Omega)} < \infty\},\ \ \|\varphi\|_{L^p(\Omega)} = (\int_\Omega |\varphi(x)|^p)^{1/p}$$
$$L^\infty(\Omega) = \{\varphi : \Omega \to \mathbb{R}; \|\varphi\|_{L^\infty(\Omega)} < \infty\},\ \ \|\varphi\|_{L^\infty(\Omega)} = \inf\{c : |\varphi(x)| \leq c \text{ a.e. } \Omega\}$$

Analogously we define subspaces where the weak derivatives up to order k are Lebesgue measurable as well.

$$W^{k,p}(\Omega) = \{\varphi : \Omega \to \mathbb{R}; \|\varphi\|_{W^{k,p}(\Omega)} < \infty\},\ \ \|\varphi\|_{W^{k,p}(\Omega)} = (\sum_{0\leq|\pi|\leq k} \|D^\pi\varphi(x)\|^p_{L^p(\Omega)})^{1/p}$$
$$W^{k,\infty}(\Omega) = \{\varphi : \Omega \to \mathbb{R}; \|\varphi\|_{W^{k,\infty}(\Omega)} < \infty\},\ \ \|\varphi\|_{W^{k,\infty}(\Omega)} = \sum_{0\leq|\pi|\leq k} \|D^\pi\varphi(x)\|_{L^\infty(\Omega)}$$

which are also Banach spaces. For $p \in [1, \infty]$, $L^p(\Omega)$ consist of functions that are defined almost everywhere in the domain and consequently do not allow for the formulation of any boundary conditions at all. However, for $k \geq 1$ we can formulate such conditions in the sense of traces ([2, 26]). In particular, we are interested in the subspaces of $W_0^{k,p}(\Omega)$ which consist of functions being zero on $\partial\Omega$ in the sense of traces. Those spaces are denoted by $W_0^{k,p}(\Omega)$ and the closure of $C_c^\infty(\Omega)$ in the corresponding norm $\|\cdot\|_{W^{k,p}(\Omega)}$. We focus on the particular choice $p = 2$. In this case, the corresponding Banach spaces are Hilbert spaces equipped with an inner product and denoted by

$$W^{k,2}(\Omega) = H^k(\Omega) \text{ with scalar product } \langle v, w\rangle_{H^k(\Omega)} = \sum_{0\leq|\pi|\leq k} \int_\Omega (D^\pi v(x))(D^\pi w(x))$$

and $H_0^k(\Omega)$ in the case of functions vanishing on the boundary in the sense of traces respectively. For the special case $k = 1, p = 2$ and homogeneous boundary conditions in the sense of traces we utilize the following equivalence of norms frequently and without further noting.

$$\sum_{|\pi|=1} |D^\pi \varphi|^2 \le \sum_{|\pi|\le 1} |D^\pi \varphi|^2 \le c \sum_{|\pi|=1} |D^\pi \varphi|^2$$

We will always refer to the norms and inner products of $L^2(\Omega)$ and $H_0^1(\Omega)$ in the following way.

$$\|\cdot\|_{L^2(\Omega)} = |\cdot|, \ \langle v_1, v_2\rangle_{L^2(\Omega)} = (v_1, v_2) \text{ and } \|\cdot\|_{H_0^1(\Omega)} = \|\cdot\|, \ \langle v_1, v_2\rangle_{H_0^1(\Omega)} = \langle v_1, v_2\rangle$$

When considering problems involving first order differential operators one is interested in function spaces that are more regular than Lebesgue spaces since first order derivatives are not defined in a weak form for elements of $L^p(\Omega)$, and less than $W^{1,p}(\Omega)$ as these spaces are already solutions spaces for second order problems and first order differential operators can be seen as degenerate instances of such. Therefore we introduce the scalar operator

$$\mathbf{b}\cdot\nabla = \sum_{i=1}^{n} b^i(x)\frac{\partial}{\partial x_i}$$

where $\mathbf{b}$ is a given, bounded and continuous vector field on $\overline{\Omega}$ satisfying

$$b^i \in C(\overline{\Omega}) \cap L^\infty(\Omega) \text{ for } i = 1, .., n \text{ and } \nabla\cdot\mathbf{b} \in L^\infty(\Omega).$$

The function space

$$L_B^2(\Omega) = \{\varphi \in L^2(\Omega) : \mathbf{b}\cdot\nabla\varphi \in L^2(\Omega)\}$$

depending on the vector field $\mathbf{b}$ meets the requirements formulated above. The following Lemma briefly presents essential properties collected from [56, 133, 135].

Lemma 2.1.2. *For $L_B^2(\Omega)$ the following holds.*

1. *$L_B^2(\Omega)$ is a Hilbert space for the graph norm*

$$\|\varphi\|_{L_B^2(\Omega)} = (|\varphi|^2 + |\mathbf{b}\cdot\nabla\varphi|^2)^{1/2}.$$

2. *$C^\infty(\overline{\Omega})$ and $C^{0,1}(\overline{\Omega})$ are dense in $L_B^2(\Omega)$*
3. *The trace mapping $\mathcal{R} : \varphi \mapsto (\mathbf{b}\cdot\mathbf{n})\varphi$ on $C^\infty(\overline{\Omega})$ admits an extension as a linear and continuous map from $L_B^2(\Omega)$ to $H^{-1/2}(\partial\Omega)$*
4. *For $\varphi \in L_B^2(\Omega)$, $v \in H^1(\Omega)$ Greens formula*

$$\int_\Omega v\mathbf{b}\cdot\nabla\varphi + \varphi\mathbf{b}\cdot\nabla v + v\varphi(\nabla\cdot\mathbf{b}) = \int_{\partial\Omega} (\mathbf{b}\cdot\mathbf{n})\varphi v$$

   holds where the surface integral is understood in the duality sense $\langle\cdot,\cdot\rangle_{H^{-1/2}(\Omega),H^{1/2}(\Omega)}$

5. *For $v_1, v_2 \in L_B^2(\Omega)$ and $\max(v_1, v_2)$ denoting the pointwise maximum we have*

$$\max(v_1, v_2) \in L_B^2(\Omega), \quad \mathbf{b}\cdot\nabla\max(v_1, v_2) = \mathbf{b}\cdot\nabla v_1\chi_{\{v_1\ge v_2\}} + \mathbf{b}\cdot\nabla v_2\chi_{\{v_2>v_1\}}.$$

2.1.4 Lebesgue Spaces for Evolutionary Problems

The function spaces introduced in this section are discussed widely in literature. We refer to [8, 157] for a detailed overview. Consider the Gelfand triple of Hilbert spaces

$$V \subset H \equiv H^* \subset V^* \tag{2.1}$$

where both embeddings are continuous and the pivot space H is identified with his dual space due to the Riesz Representation Theorem. In the scope of this work we will use $V = H_0^1(\Omega)$, $H = L^2(\Omega)$ exclusively. Thus $V^* = H^{-1}(\Omega)$. In this setting the first embedding is even compact under mild assumptions concerning the domain Ω (see [2]). The Lebesgue spaces of vector valued functions also known as Bochner spaces are defined as

$$\mathfrak{V} = L^2(0,T;V) \text{ with norm } \|v\|_{\mathfrak{V}}^2 = \int_0^T \|v(t)\|^2 dt$$
$$\mathfrak{H} = L^2(0,T;H) \text{ with norm } |v|_{\mathfrak{H}}^2 = \int_0^T |v(t)|^2 dt$$

Since V and H are Hilbert spaces, the same holds true for $\mathfrak{V}$ and $\mathfrak{H}$ with scalar products

$$(u,v)_{\mathfrak{H}} = \int_0^T (u(t),v(t))dt \quad \text{and} \quad \langle u,v\rangle_{\mathfrak{V}} = \int_0^T \langle u(t),v(t)\rangle dt.$$

The dual space of $\mathfrak{V}$ is given by (see [157])

$$\mathfrak{V}^* = L^2(0,T;V^*) \quad \text{with duality pairing} \quad \langle v,u\rangle_{\mathfrak{V}^*,\mathfrak{V}} = \int_0^T \langle v(t),u(t)\rangle_{V^*,V} dt$$

and we obtain the so called evolution triplet

$$\mathfrak{V} \subset \mathfrak{H} = \mathfrak{H}^* \subset \mathfrak{V}^*$$

with continuous embeddings. Note that we have $L^2(0,T;H) = L^2(\mathcal{Q})$ (see, e.g. [157]). We will use the same notation for the norm and scalar products of the Bochner spaces as for the underlying spaces without any indexing if it is clear which one has to be considered. Evolutionary problems include, in addition to differential operators acting on spatial variables, derivatives with respect to time. Instead of using the Gâteaux or Fréchet derivative, the operator D_t has to be understood as distributional derivative in the following sense.

Definition 2.1.1. *Let $u \in \mathfrak{V}$ be given. The function $w \in \mathfrak{V}^*$ is called the* ***distributional derivative of** u **with respect to** t if*

$$\int_0^T \varphi_t(s)(u(s),v)ds = (-1)\int_0^T \varphi(s)\langle w(s),v\rangle ds$$

holds for all $\varphi \in C_c^\infty(0,T)$ and $v \in V$.

This definition is already suited to the case considered in this thesis. The concept of generalized time derivatives is wider and explained in detail in [157]. In the same reference the following result can be found showing, that generalized time derivatives behave well under weak convergence.

Proposition 2.1.2. *Let Y and Z denote two Banach spaces with continuous embedding $Y \to Z$. If we have $y_n \rightharpoonup y$ in $L^2(0,T;Y)$ and $D_t y_n \rightharpoonup v$ in $L^2(0,T;Z)$ then*

$$v = D_t y$$

is satisfied.

Regarding the existence of solutions to partial differential equations of evolution type,

$$W(0,T) := \{v | v \in \mathfrak{V}, D_t v \in \mathfrak{V}^*\}$$

plays an important role. It is a Hilbert space with inner product

$$(u,v)_{W(0,T)} = \int_0^T \langle u(t), v(t)\rangle_V + \langle D_t u(t), D_t v(t)\rangle_{V^*} dt \text{ for all } u, v \in W(0,T)$$

and corresponding norm $\|v\|_{W(0,T)} = (v,v)^{1/2}_{W(0,T)}$. Note, that the Hilbert space nature of $W(0,T)$ is a consequence of our choice of V and H and hence not the most general setting. $W(0,T)$ is well studied and we collected several embedding properties of the space in Lemma A.2.1. We end the discussion on $W(0,T)$ with mentioning the following properties allowing to deal with the generalized time derivative.

Lemma 2.1.3 ([157])**.** *For all $v, u \in W(0,T)$ and $0 \le t \le s \le T$ we have*

$$(v(s), u(s))_H - (v(t), u(t))_H = \int_t^s \langle D_t v(\tau), u(\tau)\rangle + \langle D_t u(\tau), v(\tau)\rangle d\tau$$

especially providing

$$(1/2)(v(s), v(s))_H - (1/2)(v(t), v(t))_H = \int_t^s \langle D_t v(\tau), v(\tau)\rangle d\tau$$

In addition to $W(0,T)$ we introduce the following function space which is useful in the setting of first order differential operators.

$$\hat{W}(0,T) := \{v | v \in \mathfrak{V}, D_t v \in \mathfrak{H}\}$$

It has been considered for example in [104] and is a Hilbert space with inner product

$$(u,v)_{\hat{W}(0,T)} = \int_0^T \langle u(t), v(t)\rangle_V + (D_t u(t), D_t v(t))_{L^2(\Omega)} dt \text{ for all } u, v \in \hat{W}(0,T)$$

where the embedding

$$\hat{W}(0,T) \to W(0,T)$$

is continuous due to the continuous embedding $\mathfrak{H} \to \mathfrak{V}^*$. Finally

$$W_0(0,T) = \{\varphi \in W(0,T) : \varphi(0) = 0\}$$
$$\hat{W}_0(0,T) = \{\varphi \in \hat{W}(0,T) : \varphi(0) = 0\}$$

are closed linear subspaces of $W(0,T)$ and $\hat{W}(0,T)$ respectively.

2.2 Partial Differential Equations of Second Order

As before, we distinguish stationary and evolutionary problems.

2.2.1 Problem Formulation

In both cases, PDE's can have several levels of nonlinearity determined by the differential operator with respect to the spatial variables. If the operator is linear in the function and its derivatives up to order two, they are called linear. Thus linear differential operators of second order are always of the form

$$\mathcal{A}y = \sum_{i,j=1}^{n} a_{i,j} y_{x_i x_j} + \sum_{i=1}^{n} a_i y_{x_i} + ay \tag{2.2}$$

where the coefficients can depend on the domain of y ($x \in \Omega$ or $(x,t) \in \mathcal{Q}$).
If they are linear with respect to the second order derivatives and nonlinear with respect to the function itself as well as the first derivatives, the problem is called semilinear. As examples we mention the simplified Ginzburg-Landau-model of superconductivity as studied in [86] and the problem occuring in this thesis (see Chapter 5). Note that the usual setting for optimal control of semilinear partial differential equations only considers a nonlinearity acting on y and linear functions acting on occuring first and second derivatives. This stems from existence theory for solutions of control problems and is connected to weak and strong convergence of infimizing sequences.
If the coefficients of the linear operator acting on the highest derivatives of the solution depends on the argument and its derivatives up to order 1 the problem class is called quasilinear where in addition one distinguishes between operators with principle part in divergence form and general quasilinear ones. Finally, there are fully nonlinear partial differential equations, where the differential operator is a nonlinear function depending on the solution as well as all of its derivatives up to order 2.

Elliptic Partial Differential Equations

For elliptic problems we will restrict ourselves to the case of linear boundary value problems. The Dirichlet problem is given as

$$\begin{aligned} \mathcal{A}(x)y(x) &= f(x) \text{ in } \Omega \\ y(x) &= g(x) \text{ on } \partial\Omega \end{aligned} \tag{2.3}$$

where the differential operator $\mathcal{A}$ is assumed to satisfy the uniform ellipticity condition

$$\sum_{i,j=1}^{n} a_{i,j}(x)\xi_i\xi_j \geq \nu|\xi|^2 \tag{2.4}$$

for all $x \in \Omega$, $\xi \in \mathbb{R}^n$ and some $\nu > 0$. Moreover, f represents a volume force and g prescribes the behavior of the solution on the boundary of the domain Ω.
Note that different boundary conditions can be imposed providing for example the so called Neumann problem.

Parabolic Partial Differential Equations

Parabolic differential operators are of second order with respect to the spatial variables and contain a distributional derivative of the unknown function with respect to a parameter. They typically describe the evolution of an initial state over time obeying certain governing laws as for example in the heat equation. The linear initial value boundary value or Cauchy problem is given as

$$\begin{aligned} y_t(x,t) + \mathcal{A}(x,t)y(x,t) &= f(x,t) \text{ in } \mathcal{Q} \\ y(x,0) &= u_0(x) \text{ in } \Omega \times \{0\} \\ y(x,t) &= \Psi(x,t) \text{ on } \partial\Omega \times I \end{aligned} \tag{2.5}$$

where the differential operator $\mathcal{A}$ again is assumed to satisfy condition (2.4) for all $(x,t) \in \Omega \times I$. Besides this linear problem we consider a quasilinear parabolic partial differential equation with principle part in divergence form. In this case the unknown function has to satisfy

$$y_t(x,t) - \sum_{i=1}^{n} \frac{d}{d_i} a_i(x,t,y,Dy) + a(x,t,y,Dy) = 0 \text{ in } \mathcal{Q}. \tag{2.6}$$

The problem studied in this thesis is a further simplification given by the semilinear equation

$$y_t(x,t) - \varepsilon\Delta y + a(x,t,y,Dy) = 0 \tag{2.7}$$

with the same initial and boundary conditions on the parabolic boundary Γ as in (2.5) and $\varepsilon > 0$.
In case of a linear differential operator $\mathcal{A}$, (2.4) has to hold for all $(x,t) \in \mathcal{Q}$ whereas for nonlinear differential operators the uniform ellipticity condition is satisfied if we find

$$\sum_{i,j=1}^{n} a_{i,j}(x,t,y,0)\xi_i\xi_j \geq \nu|\xi|^2 \tag{2.8}$$

to hold for all $(x,t) \in \mathcal{Q}$, finite $y \in \mathbb{R}^1$, $\nu > 0$ and $\xi \in \mathbb{R}^n$.
A differential operator $\mathcal{A} : X \supset M \to Y$ between Banach spaces X, Y is called monotone if

$$\langle \mathcal{A}(v) - \mathcal{A}(u), v - u \rangle_{Y,X} \geq 0$$

holds for all $u, v \in M$ and strictly monotone if the inequality is satisfied strictly for all $u \neq v$.
Finally an operator is called bounded if

$$\|\mathcal{A}v\|_Y \leq c\|v\|_X$$

holds.

2.2.2 Continuous Solutions

In this part we discuss the existence of classical solutions for parabolic second order PDE's. Classical solutions refer to continuous solutions of the Cauchy problem being at least in $C^{2,1}(\mathcal{Q})$. For the elliptic case we refer to [57] where questions concerning existence, continuous dependency with respect to data and further properties are given.

The Linear Case

For linear parabolic partial differential equations we refer to the textbook [110]. The main existence result concerning classical solutions of linear parabolic equations is given as follows.

Theorem 2.2.1. *Let $\partial\Omega$ be uniformly $C^{2+\theta}$ with $0 < \theta < 1$ and $a_{i,j}, a_i, a, f \in C^{\theta,\theta/2}(\mathcal{Q})$, $\Psi \in C^{2+\theta,1+\theta/2}(\partial\Omega \times [0,T])$, $u_0 \in C^{2+\theta}(\bar{\Omega})$ be such that*

$$\Psi(0,x) = u_0(x), \quad \Psi_t(0,x) = \sum_{i,j=1}^{n} a_{i,j}(x,0)u_{0_{x_ix_j}} - \sum_{i=1}^{n} a_i(x,0)u_{0_{x_i}} - a(x,0)u_0 + f(x,0).$$

Let moreover the ellipticity condition (2.8) *be satisfied. Then* (2.5) *has a unique solution y belonging to $C^{2+\theta,1+\theta/2}(\mathcal{Q})$ and satisfying*

$$\|y\|_{C^{2+\theta,1+\theta/2}(\mathcal{Q})} \le c(\|u_0\|_{C^{2+\theta}(\bar{\Omega})} + \|f\|_{C^{\theta,\theta/2}(\mathcal{Q})} + \|\Psi\|_{C^{2+\theta,1+\theta/2}(\partial\Omega\times[0,T])})$$

for some constant c depending on the data.

The named reference also discusses the importance of Hölder continuity of the involved data.

The Semilinear Case

After discussing the linear case we provide results allowing for the estimation of several quantities of the solution to a quasilinear equation. The first one is from [98] and establishes an estimate for the norm of the solutions to (2.5) and (2.6).

Proposition 2.2.1. *Let $y \in C^{2,1}(\mathcal{Q})$ be a classical solution of* (2.7) *in $\mathcal{Q}$. Suppose that the functions $a_{i,j}(x,t,y,p)$ and $a_i(x,t,y,p)$ have finite values for any $(x,t) \in \mathcal{Q}$ and the operator satisfies the uniform ellipticity condition. Moreover, let*

$$ya(x,t,y,0) \ge -\beta_1 y^2 - \beta_2 \tag{2.9}$$

be satisfied with nonnegative constants β_1 and β_2. Then

$$\max_{\mathcal{Q}} |y(x,t)| \le \inf_{\lambda>\beta_1} e^{\lambda T} \max\{max_\Gamma |y|, (\beta_2/(\lambda-\beta_1))^{1/2}\}$$

Remark 2.2.1. *Note, that there are results available that allow for an a priory estimate of*

$$\max_{\mathcal{Q}}\{|\nabla y|\}$$

in the whole domain as well as

$$\langle \nabla y \rangle_x^\theta, \langle \nabla y \rangle_z^\theta.$$

The estimates only depend on the constants given in the following Theorem 2.2.2.. We will not repeat the results but refer to the sub chapters [98, 6.3] and [98, 6.5] where the bounds have been established.

Given the above estimates, we provide the existence theorem for classical solutions of quasilinear parabolic partial differential equations. It can be found in [97, 98] and is already suited to the semilinear equations utilized in this thesis. For the results in their complete formulation we refer to the already mentioned textbooks.

Theorem 2.2.2. *Let the following conditions hold.*

1. (2.9) *is satisfied.*

2. *For $(x,t) \in \mathcal{Q}$, $|y| \leq c$ and $\xi \in \mathbb{R}^n$ arbitrary the function $a(x,t,y,\xi)$ is continuous and the estimate*
$$\varepsilon\left(\sum_{i=1}^{n} |\xi_i|\right)(1+|\xi|_{\mathbb{R}^n}) + |a(x,t,y,\xi)| \leq r_2(1+|\xi|_{\mathbb{R}^n})^2$$
is satisfied with $r_2 > \varepsilon$. Here c denotes the upper bound from Proposition 2.2.1.

3. *For $(x,t) \in \overline{\mathcal{Q}}$, $|y| \leq c$ and $|\xi|_{\mathbb{R}^n} \leq c_1$ the function $a(x,t,y,\xi)$ is continuous and satisfies a Hölder condition in x,t,y,ξ with exponents $\beta, \beta/2, \beta$ and β respectively. Moreover, is has the partial derivatives $\partial a/\partial \xi_i$ and*
$$\max\{|\partial a/\partial \xi_i|, |(a(x,t+h,y,\xi) - a(x,t,y,\xi))/h|\} \leq c_2$$
holds. Here the upper bound c_1 denotes the gradient estimate in Remark 2.2.1.

4. *The functions describing the boundary conditions satisfies*
$$\Psi_t - \varepsilon \Delta u_0 + a(x,t,u_0,\nabla u_0) = 0$$
on S.

5. *$\partial\Omega \in C^{2+\theta}$, i.e. the boundary is locally the graph of a twice differentiable function with Hölder continuous second derivative and corresponding exponent θ.*

Then there exists a unique solution
$$y \in C^{2+\theta,1+\theta/2}(\mathcal{Q})$$
of (2.6). *In addition $y_{xt} \in L^2(\mathcal{Q})$.*

The proof is based on the Schauder fixed point Theorem (see e.g. [156]). We refer to [98, Chapter 6] for it. The result is even more general since one can obtain the existence of second order quasilinear parabolic equations with principle parts in divergence form. Then the assumptions of Theorem 2.2.2 are way more involved and we decided to present the result already suited to the case we will consider in this thesis. For example this allows for dropping the requirement, that for any $|y| \leq c$, $(x,t) \in \mathcal{Q}$ and $p \in \mathbb{R}^n$ arbitrary the estimates
$$\mu_1 |\xi|^2_{\mathbb{R}^n} \leq \sum_{i=1}^{n}\sum_{j=1}^{n} \partial a(x,t,y,p)/\partial p_i \xi_i \xi_j \leq \mu_2 |\xi|^2_{\mathbb{R}^n}$$
are fulfilled with $\mu_1 > 0$, μ_2 being finite and for all $\xi \in \mathbb{R}^n$. This has, for example, not been considered in [136].

2.2.3 Weak Solutions

Here we discuss the existence of weak solutions to (2.5) and certain instances of (2.6).

The Linear Case

We will focus on weak solutions in the Hilbert spaces we have introduced in the preceding part. The following result is a direct consequence of the Lax Milgram Lemma ([52]) for the Dirichlet boundary conditions $g \equiv 0$.

Theorem 2.2.3. *Consider a bounded linear elliptic operator $\mathcal{A} : H_0^1(\Omega) \to H^{-1}(\Omega)$. For any $f \in V^*$ there exist a unique solution $y \in H_0^1(\Omega)$ of (2.3). Moreover, there exist a constant c independent of f such that*

$$\|y\| \leq c\|f\|_{H^{-1}(\Omega)}$$

If the boundary of the domain Ω and the coefficients of the linear operator (2.2) are sufficiently regular, the solution of any elliptic partial differential equation gains regularity for $L^2(\Omega)$ data.

Theorem 2.2.4 ([52])**.** *Let $\Omega \subset \mathbb{R}^n$ be either convex or have a boundary, that is locally the graph of a Lipschitz continuous function. Assume further, that $a_{i,j} \in C^1(\bar{\Omega})$, $a_i \in L^\infty(\Omega)$, $a \in L^\infty(\Omega)$ for $1 \leq i, j \leq n$ and $f \in L^2(\Omega)$.*
Then the solution of (2.3) fulfills $y \in V \cap H^2(\Omega)$, and there exist a constant $c > 0$ with

$$\|y\|_{H^2(\Omega)} \leq c(|f| + |y|)$$

where c depends on Ω and the coefficients of $\mathcal{A}$.

Considering (2.5) we present the following result. Note that $\Psi \equiv 0$ is assumed to hold on $\partial\Omega \times I$ in this case.

Theorem 2.2.5 ([52, 157])**.** *Let $\mathcal{A} : V \to V^*$ be a linear, continuous and strongly monotone operator for the evolution triplet (2.1). Moreover, let $u_0 \in L^2(\Omega)$ and $f \in \mathfrak{V}^*$ be given. Then there exist a unique weak solution $y \in W(0,T)$ satisfying*

$$\|y\|_{W(0,T)} \leq c(|u_0|_{L^2(\Omega)} + \|f\|_{\mathfrak{V}^*})$$

where c does not depend on f and u_0.

This Theorem usually needs a certain basis of the spaces V and H to provide the boundedness of the Galerkin approximation of the solution, which is the key tool in this proof, in the corresponding function spaces. For our choice, such a basis exist as it was mentioned for example in [52]. Moreover, the result holds true under weaker assumptions as the required strong monotonicity can be replaced by the Garding inequality (see [157]).

The Quasilinear Case

Similar to the result in the case of continuous solutions one can establish the existence of unique weak solutions in case of quasilinear parabolic equations with principle part in divergence form. Since we not use such results in this thesis we only refer to [57] for the elliptic and [98, 149] for the parabolic case.

2.3 Variational Inequalities

In this section we introduce stationary variational inequalities in Hilbert spaces with second order differential operators. We formulate central existence results for solutions and introduce the equivalence to complementarity problems. Furthermore, we briefly consider the evolutionary case while shifting the existence theory for solutions to Chapter 7. Variational inequalities in function spaces have been studied extensively over the last decades. We refer to the monographs [8, 58, 94, 133] for further information on this topic. For their finite dimensional counterpart we refer to [53, 111]. Consider a Hilbert space V with norm $\|\cdot\|_V$ and scalar product $\langle\cdot,\cdot\rangle_V$. Moreover, let V^* denote its dual space and $\langle x', x\rangle_{V^*,V}$ the duality pairing. Let

$$a : V \times V \to \mathbb{R}$$

be a bilinear form on V. We introduce a linear, coercive and bounded operator

$$\mathcal{A} : V \to V^*.$$

The bilinear form

$$a(v_1, v_2) = \langle \mathcal{A}v_1, v_2\rangle_{V^*,V}$$

is coercive and bounded due to the properties of $\mathcal{A}$. Conversely, any bounded and coercive bilinear form $a(v_1, v_2)$ defines a linear bounded and coercive operator $\mathcal{A} : V \to V^*$ by

$$\langle \mathcal{A}v_1, v\rangle = a(v_1, v)$$

for all $v_1 \in V$ (see for example [94]). Given some $f \in V^*$ an abstract variational inequality (also known as variational inequality of the second kind) is defined by a lower semicontinuous, proper convex function $\varphi : V \to \bar{\mathbb{R}}$. Here proper means, that there exists at least one $v \in V$ with $\varphi(v) < \infty$. The problem is to find $y \in V$ satisfying

$$\langle \mathcal{A}y, v - y\rangle + \varphi(v) - \varphi(y) \geq \langle f, v - y\rangle \text{ for all } v \in V \text{ or equivalently } \mathcal{A}y + \partial\varphi(y) \ni f$$

where $\partial\varphi \subset V \times V^*$ denotes the subdifferential of φ. For the claimed equivalence see [8]. A detailed introduction to the topic of subdifferentials can, for example, be found in [38]. Let $\mathbf{K} \subset V$ be a closed and convex set. For the particular choice of $\varphi = \chi_{\mathbf{K}}$, the indicator function of the set $\mathbf{K}$, we obtain the variational inequality of first kind,

$$\text{find } y \in \mathbf{K}, \ \ a(y, v - y) \geq \langle f, v - y\rangle_{V^*,V} \text{ for all } v \in \mathbf{K}$$

or equivalently

$$\text{find } y \in \mathbf{K}, \ \ \langle \mathcal{A}y, v - y\rangle \geq \langle f, v - y\rangle_{V^*,V} \text{ for all } v \in \mathbf{K} \tag{2.10}$$

The existence of unique solutions is given as follows.

Proposition 2.3.1 (e.g. [94]). *Given any $f \in V^*$,* (2.10) *admits a unique solution. Moreover, the solution depends Lipschitz continuously on the data, i.e.*

$$\|y_1 - y_2\| \leq \frac{1}{c}\|f_1 - f_2\|_{H^{-1}(\Omega)}$$

holds for the coercivity constant c of the bilinear form.

Remark 2.3.1. *If $a(\cdot,\cdot)$ is symmetric or equivalently the defining linear operator $\mathcal{A}$ is self adjoint, solving* (2.10) *is equivalent to finding the unique solution of the quadratic minimization problem*

$$\min \mathcal{J}(y) = \frac{1}{2}a(y,y) - \langle f, y\rangle_{V^*,V} \ \textit{s.t.}\ \ y \in \mathbf{K}.$$

We will discuss this fact in detail in Chapter 3.

Fix $V = H_0^1(\Omega)$ and $V^* = H^{-1}(\Omega)$ and the Gelfand triple

$$V \subset L^2(\Omega) \equiv (L^2(\Omega))^* \subset V^*$$

where we identified the pivot Hilbert space with its dual space according to the Riesz representation Theorem. Similar to Theorem 2.2.4 we can establish higher regularity of the solution to (2.10) in suitable settings. Let

$$\mathbf{K} = \{\varphi \in V : v \geq \psi \text{ a.e. in } \Omega\} \tag{2.11}$$

be the convex set defining the variational inequality for a given obstacle ψ with certain regularity.

Theorem 2.3.1 ([55])**.** *Let Ω be a domain such that the boundary is locally the graph of a C^2 function. Moreover, assume, that $\mathcal{A}$ is an elliptic operator and the coefficients of $\mathcal{A}$ are elements of $L^\infty(\Omega)$. Finally, let the coefficients $a_{i,j}$ be Lipschitz continuous in Ω.*
If $f \in L^p(\Omega)$ and $\psi \in W^{2,p}(\Omega)$ for some $p \in (1,\infty)$, then the solution y of (2.10) *is an element of $W^{2,p}(\Omega)$. Moreover, the variational inequality is equivalent to the complementarity problem*

$$\mathcal{A}y - f \geq 0, \ \ y \geq \psi, \ \ (\mathcal{A}y - f)(y - \psi) = 0$$

The equivalence to a complementarity system can be established under less restrictive assumptions on the domain Ω for $p = 2$. Estimates ensuring the regularity of the image $\mathcal{A}y \in L^2(\Omega)$ are available for example in [133]. Moreover, if $a_i \equiv 0$ holds for the coefficients of $\mathcal{A}$ acting on the weak first order derivatives of y and $a_{i,j} \in C^1(\overline{\Omega})$, the regularity assumptions concerning Ω can be replaced by assuming convexity of the domain (see, e.g. [8]) still ensuring $y \in H^2(\Omega)$. In the case of evolutionary problems we are facing the following issue. Let $\mathcal{A}$ be uniformly elliptic. We have

$$\langle D_t y + \mathcal{A}y, y\rangle_{W(0.T)^*,W(0,T)} = \langle D_t y + \mathcal{A}y, y\rangle_{\mathfrak{Y}^*,\mathfrak{Y}} \geq \frac{1}{2}(|y(T)|^2_{L^2(\Omega)} - |y(0)|^2_{L^2(\Omega)}) + c\|y\|^2_{\mathfrak{Y}}.$$

Obviously, the operator $D_t + \mathcal{A}$ need not to be coercive and standard results as Proposition 2.3.1 are not applicable. This issue is for example tackled with semigroup theory or accretive operators and discussed in [8, 25]. Corresponding existence results are provided in Chapter 7. At this point we recall the definition a generalized concept of continuity, which is important for certain results in Chapter 6 and 7.

Definition 2.3.1. *Let V denote an arbitrary separable Banach space. The operator $\mathcal{A} : V \to V^*$ is called* ***hemicontinuous*** *if the mapping*

$$\gamma \mapsto \langle \mathcal{A}(v_1 + \gamma v_2), v_3\rangle_{V^*,V}$$

is continuous for all v_1, v_2, v_3 in V.

Note that this definition of hemicontinuity stems from the textbook [105] and describes a different condition than hemicontinuity introduced for set valued maps (see [5]).

Examples

The Obstacle Problem([58, 133])
The obstacle problem considers a homogenous membrane occupying a domain $\Omega \subset \mathbb{R}^2$ with regular boundary. The membrane is equally stretched in all directions by a uniform tension and loaded by a distributed force f. Moreover, it is fixed on the boundary. Let $y : \Omega \to \mathbb{R}$ describe the displacement of the membrane with respect to a given reference level. If the membrane is fixed to the reference level on the boundary, this implies

$$y(x) = 0 \text{ on } \partial\Omega.$$

Given f, the total potential energy of the membrane is given as

$$E(y) = \int_\Omega |\nabla y|^2 - \int_\Omega fy.$$

For reasons of energy minimization of the system, the displacement of the membrane for f solves the following Dirichlet problem (see [133]).

$$\begin{aligned} -\Delta y &= f \text{ in } \Omega \\ y &= 0 \text{ on } \partial\Omega \end{aligned}$$

Now assume, that there exist a rigid body whose surface is described by the function ψ. The membrane can not be displaced below this obstacle. Hence the set of all possible displacements is defined by the set $\mathbf{K}$ from (2.11). By the principle of energy minimization, the final shape of the membrane y has to satisfy

$$E(y) \leq E(v)$$

for all admissible displacements $v \in \mathbf{K}$. It has been for example shown in [133], that finding the solution is equivalent to solving the elliptic variational inequality

$$\text{find } y \in \mathbf{K} : \int_\Omega \nabla y \cdot \nabla(v - y) - \int_\Omega f(v - y) \geq 0 \text{ for all } v \in \mathbf{K}.$$

According to Theorem 2.3.1 and the comments below, the solution $y \in H_0^1(\Omega) \cap H^2(\Omega) \subset C(\overline{\Omega})$ defines disjoint subsets $\Omega_+, \mathcal{A} \in \Omega$ with

$$\Omega_+ := \{x \in \Omega : y(x) > \psi(x)\} \quad \text{and} \quad \Omega_0 := \{x \in \Omega : y(x) = \psi(x)\}$$

called inactive (or noncoincidence) and active (or coincidence) set.
Electrochemical Machining (e.c.m.)([48, 133])
A metal part can be shaped by placing it as an anode in an electrolytic cell. Applying a potential difference across the cathode and anode, between which lies an appropriate electrolyte, causes a chemical reaction on the anode surface resulting in the removal of anode metal. This

process is called electrochemical machining. By $\Omega \subset \mathbb{R}^2$ we denote the cathode with is assumed to be bounded, simply connected with smooth boundary or surface $\partial\Omega$. The anode or original metal piece is the shrinking domain $I(t)$ with Corresponding initial configuration $I_0 = I(0) \subset \Omega$. The electrolyte occupies the region $\Lambda(t) = \Omega \backslash I(t)$ and we assume, that for each time $t \in [0, T]$, the free boundary between $\Lambda(t)$ and $I(t)$ is represented by

$$\Phi(t) = \{(x, y) \in \Omega : t - l(x, y) = F((x, y), t) = 0\}$$

for the unknown function l satisfying $l(x, y) > 0$ for all $(x, y) \in int(I_0)$. Let $\gamma = \gamma(t)$,$0 < t \leq T$ denote the potential difference across the electrodes during the machining time T for the given initial anode. The quasi-steady model of electrochemical machining consist of Laplace's equation for the potential field η and an equation describing the rate of removal of anode metal derived from Faraday's and Ohm's law. For the e.c.m. constant λ the problem is formulated as follows.
Find $\eta = \eta(x_1, x_2, t)$ and $l(x_1, x_2)$ such that the following equations hold.

$$\begin{aligned} l > 0 \text{ in } I_0 \backslash \partial I_0, \quad l &= 0 \text{ on } \Lambda(0) \cup \partial I_0 \\ \Delta\eta(t) &= 0 \text{ in } \Lambda(t) \\ \eta(t) &= 0 \text{ on } \partial\Omega, 0 < t \leq T \\ \eta(t) &= \gamma(t) \text{ on } \Phi(t) \\ \nabla\eta(t) \cdot \nabla l &= \lambda \text{ on } \Phi(t) \end{aligned}$$

Utilizing the maximum principle for η at any fixed time and the Baiocchi's like transformation

$$u(x_1, x_2, t) = \int_0^t \gamma(\tau) - \eta(x_1, x_2, \tau) d\tau$$

which was introduced in [46] for the one phase Stefan, one can reduce the problem to find some u such that for all $t \in (0, T]$ the following variational inequality is satisfied.

$$u(t) \in \mathbf{K}(t), \quad \int_\Omega \nabla u \nabla(v - u) \geq \int_\Omega f(v - u) \quad \text{for all } v \in \mathbf{K}(t)$$

Here, $\mathbf{K}(t) = \{\varphi \in H^1(\Omega) : \varphi \geq 0, \varphi = g(t) \text{ on } \partial\Omega\}$ is a closed convex set defined by $g(t) = \int_0^t \gamma(\tau)$. For details of the derivation see [48, 133].

2.4 Solution Methods

Within this section, we assume the framework of Theorem 2.3.1 or the comments below to be given. Thus, the variable $\xi \in L^2(\Omega)$ in the complementarity system admits a pointwise interpretation.

2.4.1 Primal Dual Active Set Method

Due to Theorem 2.3.1 and the comments below we find the variational inequality (2.10) subject to a differential operator with sufficiently regular coefficients and for the choices $V = H_0^1(\Omega)$

and $f \in L^2(\Omega)$ with Ω locally Lipschitz to be equivalent to the complementarity system

$$\begin{aligned} Ay - \xi &= f \\ \xi \geq 0,\ \ y \geq \psi,\ \ (\xi, y - \psi) &= 0 \end{aligned} \tag{VI}$$

where the inequalities are defined in a pointwise manner.
The primal dual active set method utilizes the complementarity condition separating Ω disjointly into the active set $\Omega_0 = \{x \in \Omega : y^*(x) = \psi(x)\}$ where the solution y^* is in contact with the obstacle and the inactive set $\Omega_+ = \{x \in \Omega : y^*(x) > \psi(x)\} \subset \{x \in \Omega : \xi^*(x) = 0\}$. If the active was known a priory the complementarity condition of (VI) would decouple into $\xi^* = 0$ almost everywhere in Ω^+ and $y^* = \psi$ almost everywhere in Ω_0. Similar to finite dimensions (see [22, 59]) the iterative active set strategy generate estimates $\Omega_0^{k+1} \subset \Omega$ based on the current iterates and reduces the problem to finding $(y^{k+1}, \xi^{k+1}) \in H_0^1(\Omega) \times H^{-1}(\Omega)$ satisfying

$$\begin{gathered} Ay^{k+1} - \xi^{k+1} = f \\ y^{k+1} = \psi \text{ on } \Omega_0^{k+1},\ \ \xi^{k+1} = 0 \text{ on } \Omega_+^{k+1} = \Omega \backslash \Omega_0^{k+1} \end{gathered}$$

The lack of regularity for (y^{k+1}, ξ^{k+1}) stems from the fact, that Ω_+^{k+1} is not in general expected to meet the requirements for $y^{k+1} \in H^2(\Omega)$. Thus the normal derivatives of y^{k+1} might have jumps along the interface of Ω_0^{k+1} and Ω_+^{k+1} (see [87] for details). Consequently $\xi^{k+1} \in H^{-1}(\Omega)$ has to hold. This lack of regularity can be tackled by approximating problems. They are derived either by reformulating the regularized complementarity function

$$\xi = \sigma \max\{0, \xi - c(y - \psi)\}$$

with $c > 0$ fixed in a suitable way (for $\sigma = 1$ we obtain a proper nonlinear complementarity function) or considering a penalization of the violation of $y - \psi \geq 0$. Both ways yield the same system to be solved. Introducing an additional shift parameter $0 \leq \bar{\lambda} \in L^p(\Omega)$ with $p > 2$ which influences the feasibility of the approximations y_γ (see [87]) the approximating problems are given as

$$\begin{aligned} Ay &= f + \xi \text{ in } V^* \\ \xi &= \max\{0, \bar{\lambda} - \gamma(y - \psi)\} \text{ in } L^2(\Omega) \end{aligned} \tag{VI_γ}$$

and suitable for the primal-dual active set method as presented in Algorithm 1. Note that the second equation provides indeed increased regularity of ξ.

Algorithm 1 PDAS for (VI_γ)

Data: $y^0, \gamma, \bar{\lambda}$, set $k = 0$
Repeat
- Set $\Omega_0^{k+1} = \{x \in \Omega : \bar{\lambda}(x) - \gamma(y^k(x) - \psi(x)) > 0\}$
- Solve
  $Ay - \chi_{\Omega_0^{k+1}}(\bar{\lambda} - \gamma(y - \psi)) = f$ for $y = y^{k+1}$
- Set $\xi^{k+1} = \chi_{\Omega_0^{k+1}}(\bar{\lambda} - \gamma(y^{k+1} - \psi))$
- Set $k = k + 1$

Until Stopping Criterion is met

It is well known, that iterates obtained by Algorithm 1 converge, for fixed $\gamma > 0$, to a solution of problem (VI_γ) (see [87]).

2.4.2 Semismooth Newton Methods

(VI_γ) represents a system of nonlinear equations and a natural approach to obtain solutions is the application of Newton like methods achieving locally superlinear convergence in general. Unfortunately the pointwise max operator is, due to the nondifferentiability 0, not Fréchet differentiable and the classical Newton method can not be used. A workaround is to utilize less restrictive concepts of differentiability still providing the desired convergence properties of a Newton method based method.

Definition 2.4.1 ([37, 72]). *Let X and Y be Banach spaces and $D \subset X$ be an open subset. A mapping $F : D \to Y$ is said to be Newton- or slant- differentiable in the open subset $D_0 \subset D$ if there exists a family of mappings $G : D_0 \to L(X, Y)$ such that*

$$\lim_{h \to 0} \frac{\|F(x+h) - F(x) - G(x+h)h\|_Y}{\|h\|_X} = 0$$

holds for every $x \in D_0$.

Being aware of the fact, that this concept is known under different names we restrict ourselves to the term Newton differentiability in the remainder. In contrast to the usual Fréchet differentiability, the generalized derivative G is evaluated at the perturbed argument $x + h$. Moreover, G is neither required nor expected to be unique since the definition mirrors the finite dimensional concept of semismoothness (see [116, 129]) in function spaces. A brief comment, putting the function space concept in perspective with the finite dimensional setting can be found in [72, page 867].
This concept is sufficient to formulate a Newton like method to find $x^* \in D$ with $F(x^*) = 0$.

Theorem 2.4.1 ([37, 72]). *Let F be Newton differentiable in an open neighborhood D_0 containing the root x^* and having a Newton derivative G.*
If $G(x)$ is invertible for all $x \in D_0$ and $\{\|G(x)^{-1}\| : x \in D_0\}$ is bounded, the semismooth Newton iteration

$$x^{k+1} = x^k - G(x^k)^{-1} F(x^k)$$

converges superlinearily to x^* provided $\|x^0 - x^*\|$ is sufficiently small.

For the calculus rules concerning Newton derivatives we refer to [37, 72, 95]. Considering (VI_γ) we define the nonlinear mapping $F : V \times L^2(\Omega) \to V^* \times L^2(\Omega)$,

$$F(y, \xi) = \begin{pmatrix} F_1(y, \xi) \\ F_2(y, \xi) \end{pmatrix} = \begin{pmatrix} Ay - f - \xi \\ \xi - \max\{0, \bar{\lambda} - \gamma(y - \psi)\} \end{pmatrix}.$$

The pointwise max operator is only Newton differentiable when considered as a mapping from $L^p(\Omega)$ to $L^q(\Omega)$ with $1 \le q < p \le \infty$ (see [72]) since the norm gap is essential for the proof. As a consequence, the original problem (VI) does not yield a Newton differentiable system when the complementarity system is replaced by the NCP function

$$0 = \xi - \max\{0, \xi - c(y - \psi)\} \text{ in } L^2(\Omega)$$

($\xi \in L^2(\Omega)$ is still contained in the max) and thus the method is applicable to the approximating problems (VI_γ) only. Following [72], the Newton derivative of F at (y^k, ξ^k) is given as

$$G(y,\xi)(h_y,h_\xi) = \begin{pmatrix} Ah_y - h_\xi \\ h_\xi + \gamma\chi_{\Omega_0^k} h_y \end{pmatrix}.$$

The semismooth Newton step $(\delta y, \delta\xi)$ is defined as solution of

$$\begin{pmatrix} A\delta y - \delta\xi \\ \delta\xi + \gamma\chi_{\Omega_0^{k+1}}\delta y \end{pmatrix} = -\begin{pmatrix} Ay^k - f - \xi^k \\ \xi^k - \max\{0, \bar\lambda - \gamma(y^k - \psi)\} \end{pmatrix}$$

which is equivalent to step 3 and 4 of Algorithm 1 with guaranteed convergence. Moreover, the algorithm converges locally at a superlinear rate by the properties of the semismooth Newton method (see also [87]).

2.4.3 Path following

Finally we have to relate the auxiliary problems (VI_γ), each of whose being efficiently solvable by Algorithm 1 for arbitrary parameter $\gamma > 0$, back to (VI). In the case of elliptic variational inequalities of the first kind the following result has been proven in [87].

Theorem 2.4.2. *Let for $\gamma > 0$ $(y_\gamma, \xi_\gamma) \in V \times L^2(\Omega)$ denote the solution of (VI_γ).*
Then $y_\gamma \to y^*$ in V and $\xi_\gamma \rightharpoonup \xi^*$ in V^* as $\gamma \to \infty$ where $(y^*, \xi^*) \in V \times L^2(\Omega)$ is the solution of (VI).

The additional regularity of ξ^* stems from the regularity theory for (VI).
This convergence properties suggest an overall algorithm for solving (VI) in the following form. Choose a sequence of $\gamma \to \infty$ and solve (VI_γ) by Algorithm 1 initializing the current step by the solution of the preceding iteration.
Besides heuristic strategies for an update of γ like increasing by the same factor each iterate (see e.g. [73]) adaptive strategies are possible in certain cases where the so called primal dual path

$$C = \{(y_\gamma, \xi_\gamma) \in V \times V^* : \gamma \in (0,\infty)\}.$$

is studied. We refer to [76] where boundedness and Lipschitz continuity have been proven and a model based on the primal-dual path value functional

$$\gamma \mapsto \frac{1}{2}a(y_\gamma, y_\gamma) - (f, y_\gamma) + \frac{1}{2\gamma}\|\max\{0, \bar\lambda - \gamma(y_\gamma - \psi)\}\|^2_{L^2(\Omega)}$$

was established for such a problem instance. (VI) was considered as optimality condition for the quadratic problem

$$\min \;\; \frac{1}{2}a(y,y) - (f,y) \text{ s.t. } y \in V, \;\; y \geq \psi \text{ almost everywhere.}$$

3 Optimization

In this section we present results concerning optimization in function spaces. We always consider an objective functional

$$\mathcal{J} : \mathcal{D} \to \mathbb{R}^1$$

of certain regularity defined a subset $\mathcal{D}$ of a Banach space X. $\mathcal{D}$ represents the so called feasible set defined by certain constraints.
Here we discuss three areas of optimization in Banach spaces, namely general constraint optimization, optimal control subject to equality constraints and optimal control subject to inequality constraints. In each case we briefly discuss the function space setting and present first order necessary optimality conditions also known as stationarity conditions.

3.1 Constrained Optimization

For a review of finite dimensional constraint nonlinear optimization we refer to the textbooks [18, 121]. The general mathematical programming problem in Banach spaces is given as

$$\begin{array}{lrl} \min & f(x) & \\ \text{s.t.} & x & \in C \\ & g(x) & \in \mathbf{K} \end{array} \tag{P}$$

where $f : X \to \mathbb{R}$ is a Fréchet differentiable real valued functional with derivative f', $C \subset X$ is a nonempty closed convex subset and the Fréchet differentiable mapping $g : X \to Y$ with derivative g' represents further constraints defined in another Banach space Y. Finally $\mathbf{K} \subset Y$ is a closed, convex and pointed cone. With

$$\begin{aligned} C(x) &= \{\lambda(c - x) | c \in C, \lambda \geq 0\} \\ \mathbf{K}(y) &= \{k - \lambda y | k \in \mathbf{K}, \lambda \geq 0\} \end{aligned}$$

we denote the conical hulls of $C - \{x\}$ and $\mathbf{K} - \{y\}$ respectively. For (P) the following first order necessary optimality condition can be proven.

Theorem 3.1.1 ([83, 149])**.** *Let the feasible set*

$$\mathcal{D} = \{x \in X : x \in C, g(x) \in K\}$$

be convex and f be Gâteaux differentiable on a superset $\hat{\mathcal{D}}$ satisfying $\mathcal{D} \subset \hat{\mathcal{D}} \subset X$.
If x^* solves (P) *the following variational inequality has to be satisfied.*

$$f'(x^*)(x - x^*) \geq 0 \text{ for all } x \in \mathcal{D}$$

If the objective is convex this is a sufficient optimality condition as well. Otherwise the result only characterizes stationary points. Note, that Gâteaux differentiability, a less restrictive

concept of differentiability than Fréchet differentiability, is sufficient for the result.
Recalling Remark 2.3.1 we consider the quadratic optimization problem

$$\min \mathcal{J}(y) = \frac{1}{2}a(y,y) - \langle f, y\rangle_{V^*,V} \text{ s.t. } y \in \mathbf{K}$$

defined on a Banach space V with a symmetric and coercive bilinear form $a : V \times V \to \mathbb{R}$, a linear functional $f \in V^*$ and a closed convex set $\mathbf{K} \subset V$. Here, the Fréchet derivative of $\mathcal{J}$ at $\bar{y}$ in direction δy is, since the bilinear form is symmetric, given as

$$\langle \mathcal{J}(\bar{y}), \delta y\rangle = a(\bar{y}, \delta y) - \langle f, \delta y\rangle$$

and the claimed equivalence follows from Theorem 3.1.1. By strict convexity of the objective and convexity of $\mathbf{K}$, the minimizer is unique. For several reasons one is interested in a system of equations to describe optimal points instead of a variational inequality. Therefore we define a *Lagrange multiplier* $y^* \in Y^*$ for (P) at the optimal $\bar{x}$ as

$$\begin{aligned}
&y^* \in K^+ = \{v \in Y^* | \langle v, k\rangle \geq 0 \text{ for all } k \in K\}\\
&\langle y^*, g(\bar{x})\rangle = 0\\
&f'(\bar{x}) - y^* \circ g'(\bar{x}) \in C(\bar{x})^+ = \{v \in X^* | \langle v, c\rangle \geq 0 \text{ for all } c \in C\}
\end{aligned}$$

The following Theorem ensures existence of a bounded set of Lagrange multipliers under a certain condition called constraint qualification.

Theorem 3.1.2 ([149, 158])**.** *Let $\bar{x}$ be an optimal solution for* (P)*.*
If the constraint qualification $g'(\bar{x})C(\bar{x}) - K(g(\bar{x})) = Y$ is satisfied, the set of Lagrange multipliers for (P) *at $\bar{x}$ is nonempty and bounded.*

3.2 Optimal Control subject to Partial Differential Equations

This section briefly summarizes results and definitions from optimal control. For a comprehensive overview we refer to [83, 106, 149] where especially the first reference also covers the finite dimensional case. In optimal control we basically split the optimization variable so far called x into a part typically denoted by u which can be influenced and the so called state $y = y(u)$ which is defined by an equality constraint $E(y,u)$. In several cases this constraint describes some physical process.
Here we focus on a basic model problem where we control solutions of a linear elliptic partial differential equation. The corresponding extensions to semilinear elliptic and parabolic linear and semilinear equations can for example be found in [83, 149]. Quasilinear elliptic and parabolic equations have for example been studied in [35] and [34] respectively. We will consider distributed controls exclusively whereas boundary control of partial differential equations is possible as well (see e.g. [83]).
The model problem of optimal control subject to the Poisson equation is given as

$$\begin{aligned}
\min \quad & \mathcal{J}(y,u)\\
s.t. \quad & y \in H_0^1(\Omega), u \in U_{ad} \subset L^2(\Omega)\\
& -\Delta y = u \text{ a.e. in } \Omega, y = 0 \text{ on } \partial\Omega
\end{aligned} \tag{P_{PDE}}$$

where U_{ad} is the set of feasible controls in addition assumed to be convex. $\mathcal{J}$ is a Fréchet differentiable, convex and coercive functional bounded from below. A typical choice is the so called tracking type functional

$$\mathcal{J}(y,u) = \frac{1}{2}|y - y^d|^2 + \frac{\beta}{2}|u|^2$$

quantifying the misfit of the state to some desired state given by data $y^d \in L^2(\Omega)$. The additional term concerning the control u is a Tikhonov regularization (see, e.g. [51]) ensuring, that controls u are elements of $L^2(\Omega)$. Moreover it prevents so called Bang-Bang behavior (see, e.g. [149]).
For any given $u \in L^2(\Omega)$, there exists a unique solution $y = y(u) \in H_0^1(\Omega)$ to the underlying Poisson equation which is even more regular if the domain Ω satisfies certain assumptions (see Theorem 2.2.3 and Theorem 2.2.4). The Dirichlet conditions on the boundary are already incorporated in the solution space. For the solution- or control to state operator $\mathcal{S} : u \mapsto y$ one can show, that it is Fréchet differentiable and the derivative at $\bar{u}$ in direction δu is given as solution p of the following linear elliptic partial differential equation.

$$-\Delta p = \delta u \text{ a.e. in } \Omega, y = 0 \text{ on } \partial\Omega$$

For (P_{PDE}) we formulate the so called reduced problem as

$$\min \quad \mathcal{J}(\mathcal{S}(u), u) \quad s.t \quad u \in U_{ad} \subset L^2(\Omega).$$

By the chain rule and Theorem 3.1.1 the following first order necessary optimality condition holds.

Proposition 3.2.1. *Let u^* be an optimal control for the model problem (P_{PDE}). Then the following variational inequality has to hold for all $u \in U_{ad}$.*

$$\langle D_y\mathcal{J}(y^*, u^*), p\rangle + (D_u\mathcal{J}(y^*, u^*), u - u^*) \geq 0, \quad y^* = \mathcal{S}(u^*), \quad p = \mathcal{S}(u - u^*) \tag{3.1}$$

As in Chapter 3.1 it is a sufficient condition for minimality if the objective is convex. An essential tool for the optimal control of partial differential equations is the so called adjoint mapping.

Definition 3.2.1. *Let $A : X_1 \to X_2$ be a bounded linear mapping from a Banach space X_1 into a Banach space X_2. The **adjoint mapping** $A^* : X_2^* \to X_1^*$ is defined by the identity*

$$\langle A^* x_2', x_1\rangle_{X_1^*, X_1} = \langle x_2', Ax_1\rangle_{X_2^*, X_2}$$

With the adjoint operator the first term of (3.1) becomes

$$\langle D_y\mathcal{J}(y^*, u^*), p\rangle = \langle D_y\mathcal{J}(y^*, u^*), \mathcal{S}(u - u^*)\rangle = (S^*(D_y\mathcal{J}(y^*, u^*)), u - u^*).$$

Since the differential operator in the Poisson equation is self adjoint, we can further simplify the term to $S^*(D_y\mathcal{J}(y^*, u^*)) = S(D_y\mathcal{J}(y^*, u^*)) = p$ and obtain an equivalent formulation of the variational inequality, namely

$$(p + D_u\mathcal{J}(S(u^*), u^*), u - u^*) \geq 0, \quad p = S(D_y\mathcal{J}(S(u^*), u^*)) \text{ for all } u \in U_{ad}.$$

This reformulation is only possible, if we can define the adjoint operator of the state equation properly. Whenever this is not possible at all, we are restricted to (3.1). Next we introduce a certain form of the feasible set.

$$\mathcal{U}_{ad} = \{\varphi \in L^2(\Omega) : u_l(x) \leq \varphi(x) \leq u_u(x) \text{ a.e. in } \Omega\}$$

for $u_u, u_l \in L^2(\Omega)$. By the choices $C = X = H_0^1(\Omega) \times L^2(\Omega)$, $Y = H^{-1}(\Omega) \times L^2(\Omega) \times L^2(\Omega)$, $K = \{0\} \times L^2(\Omega)^+ \times L^2(\Omega)^+$ and $g(y,u) = (-\Delta y - u, u_u - u, u - u_l)$ one easily checks, that the constraint qualification of Theorem 3.1.2 is met. Consequently, the set of Lagrange-multipliers is bounded and we obtain the following result.

Theorem 3.2.1. *Let u^* be an optimal control of the model problem (P_{PDE}) and the set U_{ad} be given as box constraints. Moreover, let y^* denote the optimal state.*
Then there exist Lagrange multipliers $p \in (H^{-1}(\Omega))^* = H_0^1(\Omega)$ and $\mu_u, \mu_l \in L^2(\Omega)$ such that the following system is satisfied.

$$D_y\mathcal{J}(y^*, u^*) - \Delta p = 0 \tag{3.2a}$$

$$D_u\mathcal{J}(y^*, u^*) + p - \mu_u + \mu_l = 0 \tag{3.2b}$$

$$-\Delta y^* - u^* = 0 \tag{3.2c}$$

$$\mu_u \geq 0,\ u_u - u^* \geq 0,\ (\mu_u, u_u - u^*) = 0 \tag{3.2d}$$

$$\mu_l \geq 0,\ u^* - u_l \geq 0,\ (\mu_u, u^* - u_u) = 0 \tag{3.2e}$$

Here (3.2c) is called primal equation and (3.2a) is called adjoint equation and (3.2d), (3.2e) are the complementarity conditions for the control constraints.

3.3 Optimization subject to Variational Inequalities with Second Order Differential Operators

Stationarity conditions for the optimal control of variational inequalities with second order differential operators in Hilbert spaces strongly rely on the corresponding stationarity concepts in finite dimensions widely discussed for example in [53, 111]. Moreover, we refer to [73, 95] where they have been transferred to the Hilbert space setting and a certain hierarchy of them was established. Thus we only recall the definition.

3.3.1 Model Problem

In this thesis we will only consider the optimal control of variational inequalities for a given closed and convex set $\mathbf{K} = \{v \in V : v(x) \geq 0 \text{ a.e.}\} \in V$. The objective functional $\mathcal{J}$ will always assumed to be coercive, bounded from below and weakly lower semicontinuous. Moreover it is assumed to be Fréchet differentiable. The variational inequality is defined by a differential operator

$$\mathcal{A} : H_0^1(\Omega) = V \to V^* = H^{-1}(\Omega).$$

We assume $\Omega \in \mathbb{R}^m$ with $1 \leq m \leq 3$ to be a bounded domain. For the pivot space $L^2(\Omega)$ we obtain the following chain of compact inclusions.

$$V \subset L^2(\Omega) = L^2(\Omega)^* \subset V^* \tag{3.3}$$

Under the above assumptions, the abstract model problem for the optimal control of variational inequalities of the first kind is given as

$$\begin{array}{ll} \min & \mathcal{J}(y,u) \\ \text{s.t.} & y \in \mathbf{K},\ \ \langle \mathcal{A}y - f - u, v - y\rangle_{V^*,V} \geq 0 \ \ \forall v \in \mathbf{K} \end{array} \tag{P_{VI}}$$

with a given distributed force $f \in V^*$ and control $u \in V^*$. The most common way to solve problems of this type is to reformulate them as a Mathematical Program with Complementarity Constraints (MPCC) by introducing a slack variable ξ in the following way.

$$\begin{array}{lrl} \min & \mathcal{J}(y,u) & \\ \text{s.t.} & \mathcal{A}y - f - u & = \xi \\ & \xi \geq 0,\ y \geq 0,\ \langle \xi, y\rangle_{V^*,V} & = 0 \end{array} \tag{P_{MPCC}}$$

Since $\xi \in V^*$ holds a priory, the slack in general not allows for a pointwise interpretation and the nonegativity condition has to be understood in the sense of duality pairings, i.e. as

$$\langle \xi, \varphi\rangle_{V^*,V} \geq 0 \ \ \forall \varphi \in \{v \in V | v \geq 0 \text{ a.e. in } \Omega\}$$

This case of MPCC's had been considered for example in [79, 125]. In particular, the second reference analyzes problem (P_{MPCC}) for $u \in V^*$ only and derives corresponding stationarity conditions. Much of the existing work concentrates on volume force and controls $f, u \in L^2(\Omega)$ and domains of sufficient regularity. Then Theorem 2.3.1 and the comments below ensure, that ξ has an increased regularity and allows for a pointwise interpretation almost everywhere. In this work we will concentrate on problems with this properties as well and $\xi \geq 0$ is well defined. Moreover, by standard theory for duality pairings the product of ξ and y has to be considered in the $L^2(\Omega)$ sense.
In the context of optimal control subject to variational inequalities we are facing the following problems. First the feasible set of the optimization problem is not convex. This can be seen in Figure 3.1. In addition, using the reduced problem, the solution operator $y = \mathcal{S}(u)$ is, although Lipschitz continuous (see Proposition 2.3.1), not Fréchet differentiable in general. This had been overcome by introducing weaker concepts of differentiability like conical derivatives used in [117, 119]. Leaving out the definition of conical derivatives (see [117, Theorem 3.3] for details), we restrict ourselves to some notes. Given the model problem, the solution operator of the underlying variational inequality, $y = \mathcal{S}(u)$ admits a conical derivative for any $u \in \mathcal{V}^*$ which is for any given direction $\delta u \in V^*$ the unique solution of a certain auxiliary variational inequality. Moreover, based on this concept of differentiability, we can provide a first order optimality condition similar to Theorem 3.1.1 and Proposition 3.2.1.

Proposition 3.3.1 ([119])**.** *Let $D\mathcal{J}(\mathcal{S}(u), u) \in V$ denote the conical derivative of the objective with respect to the control obtained by the chain rule.*
If u^* is an optimal control for (P_{MPCC}) the following VI has to hold for all $\delta u \in V^*$.

$$\langle D\mathcal{J}(\mathcal{S}(u^*), u^*), \delta u\rangle \geq 0$$

Proposition 3.3.1 also holds true, if we assume $u \in U_{ad} \subset L^2(\Omega)$ with U_{ad} convex and closed while the authors declare, that further results of [119] introducing a stationary system are only applicable for the case $U_{ad} = L^2(\Omega)$. This issue has been discussed recently in [153] and the corresponding results were extended to certain convex and closed sets $U_{ad} \neq L^2(\Omega)$.

3.3.2 Stationarity Concepts

Here we recall stationarity concepts for the optimal control of variational inequalities in function spaces. We introduce the following sets for any elements being feasible for (P_{MPCC}).

$$\begin{array}{rll} \Omega_+ : & \{x \in \Omega : y(x) > 0\} & \text{noncoincidence set} \\ \{\xi > 0\} : & \{x \in \Omega : \xi(x) > 0\} & \\ B : & \{x \in \Omega : y(x) = \xi(x) = 0\} & \text{biactive set} \end{array}$$

Note that the set $\{\xi > 0\}$ is only well defined since the setting we have chosen ensures $\xi \in L^2(\Omega)$ and consequently, the slack of the underlying problem admits a pointwise interpretation. By the complementarity condition, the domain Ω is disjointly separated into

$$\Omega = \Omega_+ \,\dot{\cup}\, B \,\dot{\cup}\, \{\xi > 0\}.$$

Definition 3.3.1. $(y, u, \xi) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ *is called **W-stationary** point of* (P_{MPCC}) *if there exist an adjoint state* $p \in H_0^1(\Omega)$ *and a multiplier* $\lambda \in H^{-1}(\Omega)$ *such that the following system is satisfied.*

$$D_y\mathcal{J}(y,u) - \lambda + \mathcal{A}^* p = 0 \tag{3.4a}$$

$$D_u\mathcal{J}(y,u) - p = 0 \tag{3.4b}$$

$$\mathcal{A}y - u - f - \xi = 0 \tag{3.4c}$$

$$\xi \geq 0,\ y \geq 0,\ (\xi, y) = 0 \tag{3.4d}$$

$$\forall x \in \Omega \text{ with } \xi(x) > 0 : \ \ p(x) = 0 \tag{3.4e}$$

Moreover, the multiplier has to satisfy

$$\langle \lambda, \varphi \rangle_{V^*,V} = 0 \ \ \forall \varphi \in V, \varphi = 0 \ a.e. \ in \ \Omega \backslash \Omega_+. \tag{3.5}$$

Since (3.5) *is rather restrictive we are also interested concepts weakening this requirement. The point is called **almost W-stationary** if the multiplier satisfies in addition to* (3.4)

$$\langle \lambda, y \rangle = 0$$

and

$$\langle \lambda, \varphi \rangle = 0 \ \forall \varphi \in V, \ \varphi = 0 \ a.e. \ in \ \Omega \backslash \Omega_+, \ \varphi_{|\Omega_+} \in H_0^1(\Omega_+) \tag{3.6}$$

Finally, (y, u, ξ) *is called **$\mathcal{E}$-almost W-stationary** if* (3.6) *is replaced by the following condition. For every* $\tau > 0$ *there exist a set* $E_\tau \subset \Omega_+$ *with* $|\Omega_+ \backslash E_\tau| \leq \tau$ *and*

$$\langle \lambda, \varphi \rangle = 0 \ \forall \varphi \in V, \ \varphi = 0 \ a.e. \ in \ \Omega \backslash E_\tau$$

The difference between W- and almost W-stationarity depends on the structure of the nonincidence set Ω_+. For general subsets $\Omega \subset \mathbb{R}^n$ elements of $H_0^1(\Omega)$ can only be extended to $\mathbb{R}^n$ (and thus any superset of Ω) if the boundary of Ω satisfies certain properties. In particular this holds true, if Ω_+ is a Lipschitz domain (Theorem of Calderon-Stein [30]) but the extension result also holds for more general domains as shown in [89].
Whenever it is possible, to obtain information on the behavior of the product of the multipliers

λ and p we can define the following stronger concept of stationarity.

Definition 3.3.2. *The point* $(y, u, \xi) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ *is called* ***$\mathcal{E}$-almost C-stationary, almost C-stationary, C-stationary*** *if it is $\mathcal{E}$-almost W-stationary, almost W-stationary, W-stationary and the multipliers satisfy*

$$\langle \lambda, p \rangle \leq 0$$

Finally, we introduce the concept of strong stationarity.

Definition 3.3.3. *Let* $(y, u, \xi) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ *be a feasible point for* (P_{MPCC}) *with associated multipliers* $(p, \lambda) \in H_0^1(\Omega) \times H^{-1}(\Omega)$ *such that* (3.4) *is satisfied. Now the triplet is called* ***S-stationary*** *if in addition*

$$\begin{aligned} p &\leq 0 \text{ a.e. in } B \\ \langle \lambda, \varphi \rangle &\geq 0 \text{ for all } \varphi \in V : \varphi(x) \geq 0 \text{ a.e. in } B, v(x) = 0 \text{ a.e. in } \{\xi > 0\} \end{aligned}$$

holds.
The point is called ***almost S-stationary*** *if in addition*

$$\begin{aligned} \langle \lambda, y \rangle &= 0 \\ \langle \lambda, p \rangle &\leq 0 \\ p &\leq 0 \text{ a.e. in } B \\ \langle \lambda, \varphi \rangle &\geq 0 \text{ for all } \varphi \in V : \varphi(x) \geq 0 \text{ a.e. in } B, v(x) = 0 \text{ a.e. in } \{\xi > 0\}, (-\varphi)^+_{|\Omega_+} \in H_0^1(\Omega_+) \end{aligned}$$

The point is called ***$\mathcal{E}$-almost S-stationary*** *if in addition*

$$\begin{aligned} \langle \lambda, y \rangle &= 0 \\ \langle \lambda, p \rangle &\leq 0 \\ p &\leq 0 \text{ a.e. in } B \end{aligned}$$

holds and for every $\tau > 0$ there exist a set $E_\tau \subset \Omega_+$ with $|\Omega_+ \backslash E_\tau| \leq \tau$ and

$$\langle \lambda, \varphi \rangle \geq 0 \text{ for all } \varphi \in V : \varphi(x) \geq 0 \text{ a.e. in } B, v(x) = 0 \text{ a.e. in } \Omega \backslash (\Omega_+ \cup B)$$

In [73, 95] the following hierarchy scheme of the presented stationarity concepts has been established.

$$\begin{array}{ccccc} \text{S-stationarity} & \Rightarrow & \text{almost S-stationarity} & \Rightarrow & \mathcal{E}\text{-almost S-stationarity} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{C-stationarity} & \Rightarrow & \text{almost C-stationarity} & \Rightarrow & \mathcal{E}\text{-almost C-stationarity} \\ \Downarrow & & \Downarrow & & \Downarrow \\ \text{W-stationarity} & \Rightarrow & \text{almost W-stationarity} & \Rightarrow & \mathcal{E}\text{-almost W-stationarity} \end{array}$$

All concepts strongly benefit from the function space setting for (P_{MPCC}), namely the Gelfand triple (3.3), and the nature of the second order differential operator. The idea of the concepts is motivated by the optimal control of finite dimensional variational inequalities. So λ could be understood as multiplier for the condition $y \in \mathbf{K}$ while p could be interpreted as a multiplier for the equality constraint $\mathcal{A}y - f - u - \xi$. Since multipliers are usually elements of the dual space of the image of a constraint we obtain for the setting at hand $\lambda \in V^*$ and $p \in (V^*)^* = V$. In particular this allows for the evaluation of the pairing

$$\langle \lambda, p \rangle.$$

We will consider degenerate differential operators in this thesis where the image spaces of the equality constraint is not $H^{-1}(\Omega)$ but the pivot space $L^2(\Omega)$. The multiplier λ is still an element of $H^{-1}(\Omega)$ but p is now an element of $L^2(\Omega) = (L^2(\Omega))^*$. Consequently, the product of λ and p is not longer well defined and we can not expect to obtain something that is similar to more than W-stationarity.

3.3.3 Approximation Techniques

Problems like (P_{VI}) have been studied in detail in the last years. Two methods have proven to be efficient and will be discussed briefly in this section.
The first presented approach is designed to keep the bilevel structure of the problem intact. As shown for example in [8, 58], variational inequalities can be approximated by a sequence of nonlinear partial differential equations, which are defined by a penalization operator π satisfying

$$\begin{cases} \pi : V \to V^* \text{ is Lipschitz continuous} \\ ker(\pi) = \mathbf{K} \\ \pi \text{ is monotone} \end{cases} \tag{3.7}$$

The approximating nonlinear equations are given as

$$\mathcal{A}y_\gamma + \gamma\pi(y_\gamma) = f \text{ in } V^*$$

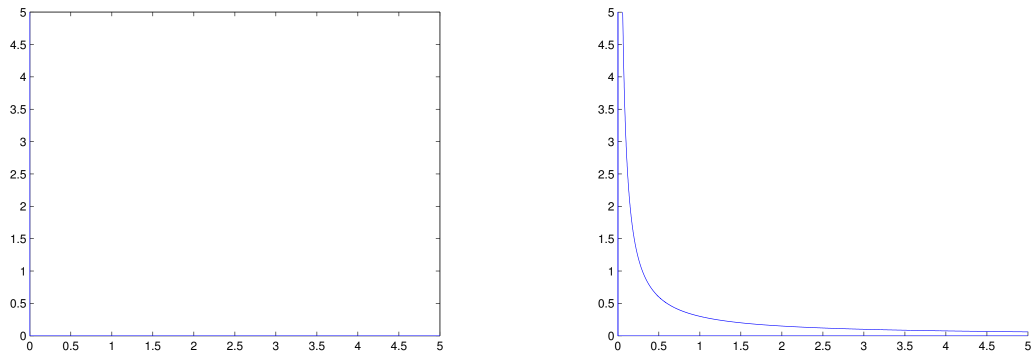
where the degree of nonlinearity (see Section 2.2) depends on the choice of π. It is well known that the corresponding solutions converge to the solution of (2.10) for $\gamma \to \infty$. It has been shown in [74] that a sequence of local solutions of the problems

$$\begin{array}{rl} \min & \mathcal{J}(y_\gamma, u_\gamma) \\ \text{s.t.} & (y_\gamma, u_\gamma) \in H_0^1(\Omega) \times L^2(\Omega) \\ & \mathcal{A}y_\gamma + \gamma\pi(y_\gamma) = f + u_\gamma \end{array} \tag{P_{VI}^γ}$$

converges, even under presence of additional control constraints, to a solution of (P_{VI}).
The second approach is based on the equivalent problem (P_{MPCC}) and aims on relaxing the complementarity constraint. In Figure 3.1 we display the feasible set of the simple complementarity constraint $0 \leq y \in \mathbb{R}^1$, $0 \leq \xi \in \mathbb{R}^1$, $y\xi = 0$ and the regularized version in the sense of [141] where the feasible set is inflated and the resulting constraint is given as $0 \leq y \in \mathbb{R}^1$, $0 \leq \xi \in \mathbb{R}^1$, $y\xi \leq \alpha$ for $\alpha > 0$. This relaxation scheme was also used in [16] in a function space setting. The family of approximating problems is given as

$$\begin{array}{rl} \min & \mathcal{J}(y_\alpha, u_\alpha) + \kappa(\alpha)|\xi_\alpha|^2 \\ \text{s.t.} & (y_\alpha, u_\alpha, \xi_\alpha) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \\ & \mathcal{A}y_\alpha - f - u_\alpha = \xi_\alpha \\ & y_\alpha \geq 0, \xi_\alpha \geq 0, (y_\alpha, \xi_\alpha) \leq \alpha \end{array} \tag{P_{MPCC}^α}$$

where $\kappa(\alpha)$ tends to zero as α does. The additional penalization term $\kappa(\alpha)|\xi_\alpha|^2$ is necessary for the following reason. In [16, Remark 3.1] we find an illustrative example, that (P^α_{MPCC}) without the penalization term has no solution in the given spaces. Instead of using an additional constraint on ξ_α as suggested in [16], we utilize the approach of [73].



(a) Feasible set for the constraint $0 \le y \in \mathbb{R}^1$, $0 \le \xi \in \mathbb{R}^1$, $y\xi = 0$

(b) Feasible set for the constraint $0 \le y \in \mathbb{R}^1$, $0 \le \xi \in \mathbb{R}^1$, $y\xi \le \alpha$

Figure 3.1: Impact of the Relaxation Technique by Scholtes

Note that there is a large variety of relaxation techniques available which has proven to be useful in the context of finite dimensional MPCC's and which might be applicable in the case at hand as well. We refer to [90, 92] and the references therein for an overview on relaxation methods for finite dimensional MPCC's and the corresponding convergence properties.
Since (P^α_{MPCC}) is still a state constrained problem, we face difficulties in connection with Lagrange multipliers which are known to have low regularity (see, e.g. [32]). This stems from the fact, that the state equation implies additional regularity of the solution, namely $y \in H^1_0(\Omega) \cap H^2(\Omega) \subset C_0(\Omega)$ for $n \le 3$. Thus the multiplier for the pointwise nonegativity condition of y has to be an element of a space space containing the regular Borel measures representing the dual space of $C_0(\Omega)$ (see [45]). To overcome this issue, a further penalization step is introduced, approximating (P^α_{MPCC}) by the family defined as

$$\begin{array}{ll} \min & \mathcal{J}(y_{\alpha,\gamma}, u_{\alpha,\gamma}) + \kappa(\alpha)|\xi_{\alpha,\gamma}|^2 + (2\gamma)^{-1}|(\bar{\lambda} - \gamma y_{\alpha,\gamma})^+|^2 \\ \text{s.t.} & (y_{\alpha,\gamma}, u_{\alpha,\gamma}, \xi_{\alpha,\gamma}) \in H^1_0(\Omega) \times L^2(\Omega) \times L^2(\Omega) \\ & \mathcal{A}y_{\alpha,\gamma} - f - u_{\alpha,\gamma} = \xi_{\alpha,\gamma} \\ & \xi_{\alpha,\gamma} \ge 0, (y_{\alpha,\gamma}, \xi_{\alpha,\gamma}) \le \alpha \end{array} \qquad (P^{\alpha,\gamma}_{MPCC})$$

for a nonegative function $\bar{\lambda} \in L^p(\Omega)$ with $p > 2$. This technique, presented in [75] and further analyzed in [77] not only ensures, that solutions of $(P^{\alpha,\gamma}_{MPCC})$ converge to solutions of (P^α_{MPCC}) while the term $(\bar{\lambda} - \gamma y_{\alpha,\gamma})^+ \in L^2(\Omega)$ converges to the multiplier of the constraint $y_\alpha \ge 0$ in (P^α_{MPCC}) in the correct space. Moreover the convergence process can be diagonalized since a careful updating of α depending on γ in the auxiliary problems provides convergence of the solutions belonging to $(P^{\alpha,\gamma}_{MPCC})$ to a solution of (P_{MPCC}) with the same beneficial approximation properties for the involved Lagrange multipliers. This has been shown in [73] for the case of elliptic and in [95] for parabolic differential operators.

4 1st Order Hyperbolic Differential Operators

Chapter 2 considered partial differential equations and variational inequalities where the involved differential operator was of second order with respect to the spatial variables. In this chapter we focus on similar objects but the differential operator is degenerated in a certain sense. Employing the notion from [108] we introduce a general superposition operator

$$H : \mathbb{R}^n \times \mathbb{R} \times \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}, \quad H \circ y = H(x, t, y, \nabla y).$$

Here the gradient represents the partial derivatives of y with respect to the spatial variables $x \in \Omega$. In case of non evolutionary problems H does not depend on t. The Dirichlet problem is defined as

$$H(x, y, \nabla y) = 0 \text{ on } \Omega, \quad y = \varphi \text{ on } \partial\Omega \tag{4.1}$$

where $\varphi : \partial\Omega \to \mathbb{R}$ is a given function. The time dependent Cauchy problem is given by

$$y_t + H(x, t, y, \nabla y) = 0 \text{ on } \mathcal{Q}, \quad y = \varphi \text{ on } \partial\Omega \times [0, T], \quad y(x, 0) = u_0(x) \text{ in } \Omega. \tag{4.2}$$

The problem is described by a given terminal time $T > 0$ and boundary and initial conditions φ and u_0 respectively. The function H is often called Hamiltonian.
As we will see in an example, we can not expect to find classical solutions as in the case of second order operators (see Theorem 2.2.2 for the time depending case). As a consequence we have to consider generalized solutions which address this difficulty but introduce a new challenge to the problem class since they are not unique in general.

4.1 Generalized Solutions

Here we briefly introduce solution concepts for first order hyperbolic PDE's. Besides viscosity solutions, which are of major interest in this thesis, we will also briefly cover the concept of entropy solutions. They are used in the context of nonlinear conservation and balance laws like the isentropic Euler equations describing the transport of gas. In addition such solutions are equivalent to viscosity solutions in the case of one dimensional domains $\Omega \subset \mathbb{R}^1$.

4.1.1 Entropy Solutions

Entropy solutions were introduced in [96]. They select the "physical relevant" generalized solution to certain first order problems (see [112] for an overview). In these problems the Hamiltonian has divergence form and thus is a specific instance of first order quasilinear differential operators. It is given as

$$H(x, t, y, \nabla y) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i} f_i(x, t, y) - g(x, t, y) = \nabla \cdot \mathbf{f}(x, t, y) - g(x, t, y). \tag{4.3}$$

In this setting the vector field $\mathbf{f} = (f_1, ..., f_n) : \mathbb{R}^n \to \mathbb{R}$ is called flux and g is called source term. To avoid the involved discussion of boundary conditions (see e.g. [11]) we restrict ourselves to $\Omega = \mathbb{R}^n$ and $\mathcal{Q}$ as already defined.

Definition 4.1.1. *A function $y \in L^\infty(\mathcal{Q})$ is called **entropy solution** of (4.2) with a differential operator of the form (4.3) if for all $c \in \mathbb{R}$ and*

$$\eta_c(\lambda) = |\lambda - c|, \quad q_{i,c}(\lambda) = sgn(\lambda - c)|f_i(\lambda) - f_i(x)|$$

the entropy inequality

$$(\eta_f(y))_t + \sum_{i=1}^{n}(q_{i,c}(y))_{x_i} \leq sgn(y-c)g(x,t,y) \ \text{in } \mathcal{D}'(\mathcal{Q}) \tag{4.4}$$

holds and the initial data $u_0 \in L^\infty(\mathbb{R}^n)$ are satisfied in the following sense

$$\lim_{t\to 0} \frac{1}{t}\int_0^t \|y(\cdot,\tau) - u_0\|_{L^1(K)} d\tau = 0 \ \text{for all compact } K \subset \mathbb{R}^n.$$

One can show, that entropy solutions are weak solutions in the sense of distributions. In addition, it can be shown (see, [96, 112]), that the entropy condition (4.4) is satisfied if y is bounded almost everywhere and the limit of solutions y^ε for $\varepsilon \to 0, \varepsilon > 0$ of the parabolic regularization

$$y_t^\varepsilon + \sum_{i=1}^{n}(f_i(y^\varepsilon))_{x_i} = g(x,t,y^\varepsilon) + \varepsilon\Delta y^\varepsilon \text{ in } \mathcal{Q}$$
$$y^\varepsilon(0,x) = u_0(x) \text{ in } \mathbb{R}^n.$$

We refer to [151] and [96] for existence, stability and uniqueness results for the case of scalar problems while for example [24] deals with systems of n equations for n unknown functions .

4.1.2 Viscosity Solutions

A further class of generalized solutions are so called *viscosity solutions*. In this section we collect major properties and results which for example can be found in [41, 108].
Considering Lipschitz continuous functions the Theorem of Rademacher (A.1.2) provides the existence of derivatives almost everywhere in the open domain Ω. Thus a natural way to define generalized solutions to (4.1),(4.2) is to consider Lipschitz continuous functions satisfying the equation pointwise almost everywhere. Note that even the set of Lipschitz continuous functions satisfying this requirement is not a singleton in general as the following examples show for the Dirichlet and the Cauchy problem. Both can be found in [108].

Example 4.1. *Let the domain $\Omega = (0,1)$ be given and consider the Dirichlet problem*

$$|\nabla y| = 1 \ \text{a.e. in } \Omega, \ \ y(0) = y(1) = 0.$$

Obviously, there can not exist a C^1 solution (by Mean Value Theorem ∇y would have to vanish *at some point in* $(0,1)$*). The function*

$$y_n(x) = \begin{cases} x - \frac{2k}{2^n} & \text{if} \quad \frac{2k}{2^2} \leq x \leq \frac{2k+1}{2^2} \\ \frac{2k+2}{2^2} - x & \text{if} \quad \frac{2k+1}{2^n} \leq x \leq \frac{2k+2}{2^n} \end{cases}$$

with $k = 0, ..., 2^n$ *is defined for every* $n \in \mathbb{N}$ *and Lipschitz continuous satisfying the partial differential equation almost everywhere.*

Example 4.2. *Consider the Cauchy problem*

$$y_t + |\nabla y| = 0 \text{ in } \mathbb{R}^n \times (0, \infty), \quad y(x,0) = 0 \text{ in } \mathbb{R}^n.$$

In this case the problem admits the classical solution $y \equiv 0$ *but it is not unique in the class of Lipschitz functions satisfying the problem almost everywhere. The function*

$$y(x,t) = \begin{cases} 0 & \text{if } |x| > t \\ |x| - t & \text{if } |x| \leq t \end{cases}$$

is Lipschitz continuous on $\mathbb{R}^n \times (0, \infty)$*, bounded for* $t \leq T$ *with arbitrary* $T > 0$ *and satisfies*

$$y_t = |\nabla y| = 0 \text{ if } |x| > t, \quad y_t = -1, \quad |\nabla y| = 1 \text{ if } |x| < t, x \neq 0.$$

According to Example 4.1, $|\nabla y| = 1$ *does not have a unique solution.*

Thus we have to select a certain function among all possible candidates.

Definition 4.1.2. *Considering the equation*

$$y_t(x,t) + H(x,t,y,\nabla y) = 0$$

A function $y \in C(\overline{\mathcal{Q}})$ *is called* ***viscosity solution*** *if for all* $\phi \in C^1(\overline{\mathcal{Q}})$ *the following holds.*

- *if* $y - \phi$ *attains a local maximum at* $(x,t)_0 \in \mathcal{Q}$*, then*

$$\phi_t(x,t)_0 + H((x,t)_0, \phi(x,t)_0, \nabla\phi(x,t)_0) \leq 0$$

- *if* $y - \phi$ *attains a local minimum at* $(x,t)_0 \in \mathcal{Q}$*, then*

$$\phi_t(x,t)_0 + H((x,t)_0, \phi(x,t)_0, \nabla\phi(x,t)_0) \geq 0$$

If only the first (second) inequality is satisfied, y *is called* ***viscosity sub- (super-)solution****.*

For Dirichlet problems such solutions are defined similar. In this case the inequalities have to hold for $H((x)_0, \phi(x)_0, \nabla\phi(x)_0)$ and $\phi \in C^1(\overline{\Omega})$. Note that there are several ways to define viscosity solutions as discussed in [40] but the reference also proves the equivalence of them. The basic idea is to put the derivatives on a test function via the maximum principle.
Note that in the case of stationary problems the sign of the defining equation matters, i.e. $H(x,t,y,\nabla y) = 0$ has a different viscosity solution from $-H(x,t,y,\nabla y) = 0$. To see this, we will briefly revisit Example 4.1.

Example 4.3 (Example 4.1 revisited)**.** *For* $[0,1]$ *the function*

$$\hat{y}(x) = -|x - 1/2| + 1/2$$

is a viscosity solution of $|\nabla y| - 1 = 0$ *in* $(0,1)$*,* $y(0) = y(1) = 0$ *but not of* $1 - |\nabla y| = 0$*.*
Take an arbitrary function $\varphi \in C^1(0,1)$ *such that* $\hat{y} - \varphi$ *has maximum (minimum) at* $x^* \in (0,1)$*.* *Since* $\hat{y}(x)$ *is differentiable everywhere except in* $x^* = 1/2$ *we obtain* $|\nabla\varphi(x)| = |\nabla y(x)| = 1$ *for all* $x \in (0,1), x \neq 1/2$*. To establish the subsolution property we assume w.l.o.g., that* $\hat{y} - \varphi$ *has a strict local maximizer in* $x = 1/2$ *satisfying* $\hat{y}(1/2) = \varphi(1/2)$*. By the triangle inequality we obtain for all* x *in a small open neighborhood around* $1/2$ *the estimate*

$$-|x| \le \hat{y}(1/2 + x) - \hat{y}(1/2) \le \varphi(1/2 + x) - \varphi(1/2)$$

providing

$$\frac{\varphi(x) - \varphi(1/2)}{x} \le 1 \text{ for } x < 0 \text{ and } \frac{\varphi(x) - \varphi(1/2)}{x} \ge -1 \text{ for } x > 0$$

Since $\varphi \in C^1(0,1)$*, we can consider the limit* $x \to 0$ *and obtain* $|\nabla\varphi(1/2)| \le 1$ *which demonstrates the subsolution property.*
For the supersolution property we use the same arguments to obtain

$$\lim_{x \to 0+} \frac{\varphi(x) - \varphi(1/2)}{x} \le -1 \text{ and } \lim_{x \to 0-} \frac{\varphi(x) - \varphi(1/2)}{x} \ge 1$$

Thus there can not exist a C^1*-function touching* $\hat{y}$ *from below in* $x = 1/2$*.*
Next consider $\varphi(x) = -(x - 1/2)^2 + 1/2$ *touching* $\hat{y}$ *from above at* $x = 1/2$ *and obtain*

$$1 - |\nabla\varphi(1/2)| = 1 > 0.$$

Consequently, $\bar{y}$ *is not a subsolution of* $1 - |\nabla y| = 0$*.*

Remark 4.1.1. *Originally, viscosity solutions are merely continuous and we can just define sub- and super solutions of the first order partial differential equations. If one can proof additional regularity, namely local Lipschitz continuity of the solution, the following holds. Due to[41, Corrolary I.6], the hyperbolic equation is satisfied at all points of differentiability of a continuous viscosity solution although this set might be empty. Every locally Lipschitz continuous function is differentiable almost everywhere on its domain. Combining the results, we find that for locally Lipschitz continuous generalized solutions, the first order equation is satisfied pointwise almost everywhere.*

We will always impose conditions ensuring Lipschitz continuity of the solutions to the first order equations.
The usual technique of selecting the viscosity solution among all generalized solutions of the first order partial differential equation is to smooth the differential operator with respect to the spatial variables by introducing an artificial viscosity term $-\varepsilon\Delta$ justifying the name. The resulting equation is, instead of a nonlinear first order partial differential equation, a nonlinear second order PDE with corresponding theory (see [97, 98]). The solutions of the regularized problems are related to viscosity solutions of the first order partial differential equation in the following way. Consider a general Hamiltonian function H defined in (4.3).

Theorem 4.1.1 ([41]). *Let $y^\varepsilon \in C^{1,2}(\overline{\mathcal{Q}})$ be a solution of*

$$\begin{aligned} y^\varepsilon_t - \varepsilon\Delta y^\varepsilon + H_\varepsilon(x,t,y^\varepsilon,\nabla y^\varepsilon) &= 0 \text{ in } \mathcal{Q} \\ y^\varepsilon &= \varphi_\varepsilon \text{ on } \partial\Omega \times [0,T] \\ y^\varepsilon(0) &= u_{0_\varepsilon} \text{ in } \overline{\Omega} \end{aligned}$$

and $y \in C(\overline{\mathcal{Q}})$. Assume $H_\varepsilon \to H$ in $C(\mathcal{Q} \times \mathbb{R} \times \mathbb{R}^n)$, $\varphi_\varepsilon \to \varphi$ in $C(\partial\Omega \times [0,T])$ and $u_{0_\varepsilon} \to u_0$ in $C(\overline{\Omega})$. If $\varepsilon \to 0$ and $y^\varepsilon \to y$ in $C(\mathcal{Q})$, then y is a viscosity solution of

$$y_t + H(x,t,y,\nabla y) = 0 \text{ in } \mathcal{Q}.$$

If the convergence $y^\varepsilon \to y$ is in $C(\overline{\mathcal{Q}})$, then the limit y also satisfies

$$y = \varphi \text{ on } \partial\Omega \times [0,T], \quad y = u_0 \text{ on } \overline{\Omega}$$

Uniqueness as well as continuous dependency of the solutions to the data is a consequence of the following result.

Theorem 4.1.2 ([108]). *Let y^1, y^2 denote Lipschitz continuous sub- and supersolutions to*

$$y^1_t + H(x,t,\nabla y^1) = 0 \quad \text{and} \quad y^2_t + H(x,t,\nabla y^2) = g$$

with $g \in C(\mathcal{Q}) \cap L^\infty(\mathcal{Q})$. Then we have

$$\|(y^1-y^2)^+\|_{L^\infty(\mathcal{Q})} \le \|(y^1-y^2)^+\|_{L^\infty(\Gamma)} + \int_0^T \|g(\cdot,s)\|_{L^\infty(\Omega)} ds$$

Setting $g \equiv 0$ and exchanging y^1 and y^2 provides uniqueness of the Lipschitz continuous generalized solution. We close the subsection by presenting a convergence result for viscosity solutions.

Theorem 4.1.3 ([40]). *Let $H_n(x,t,y,p)$ be a sequence of continuous functions such that $H_n \to H$ uniformly on compact subsets of $\mathcal{Q} \times \mathbb{R} \times \mathbb{R}^n$. Let y_n be a viscosity solution of $\partial_t y_n + H_n(x,t,y_n,Dy_n) = 0$ in $\mathcal{Q}$.*
If y_n converges uniformly to some y then y is a viscosity solution of $y_t + H(x,t,y,Dy) = 0$.

For a further discussion of this objects we refer to [108] and the references therein.
Note that viscosity solutions are used in the context of fully nonlinear second order partial differential equations as well (see, e.g. [85, 109]).

4.1.3 Entropy vs Viscosity Solutions

Both generalized solutions are obtained by the vanishing viscosity approach. It is well know, that entropy and viscosity solutions for problems of the form (4.3) are equivalent in one space dimension (see [1, 93] for details of the proofs) in certain cases. Let for $\mathcal{Q} = \mathbb{R}^1 \times (0,\infty)$, the viscosity solution y^v to the problem

$$y^v_t + H(y^v_x) = 0, y^v(x,0) = u_0$$

and the entropy solution y^e to the problem

$$y_t^e + H(y^e)_x = 0, y^e(x,0) = \frac{d}{dx}u_0$$

be given. Note that there is no source term and no spatial or temporal dependency of the flux. Then they are related according to

$$y^e = \partial_x y^v \text{ a.e..}$$

For multiple spatial dimensions, this is no longer valid (see [93] and the references therein). Further discussion on connections between classes of generalized solutions to first order PDE's can be found in [31].

4.1.4 Examples

As pointed out earlier entropy solutions are suited to physical processes such as transport of species trough domains. In particular this includes fluid dynamics or gas transport. For an overview we refer to [102]. Viscosity solutions are closely related to the optimal control of dynamical systems (see [9, 108]). In the context of deterministic optimal control problems without boundary conditions, let the state $\mathbf{y} \in \mathbb{R}^n$ be defined by a system of ODE's

$$\mathbf{y}_t = -b(\mathbf{y}(t), \mathbf{u}(t)), \ \ t \geq 0, \ \ \mathbf{y}(0) = x \in \mathbb{R}^n.$$

Here the control $\mathbf{u} \in V \subset \mathbb{R}^n$ is an element of the set of feasible controls. Under certain conditions on $b : \mathbb{R}^n \times V \to \mathbb{R}^n$ the underlying system admits a unique solution $\mathbf{y}^x$ for any given initial condition $x \in \mathbb{R}^n$. Let the pay-off functions be given as

$$\mathcal{J}(x,t;v) = \int_0^t f(\mathbf{y}^x(\tau), \mathbf{u}(\tau)) e^{-\int_0^\tau c(\mathbf{y}^x(s),\mathbf{u}(s))ds} d\tau + u_0(\mathbf{y}^x(T), \mathbf{u}(T)) e^{-\int_0^t c(\mathbf{y}^x(s),\mathbf{u}(s))ds}$$

for data f, c, u_0 satisfying certain conditions. For any (x,t) we define

$$w(x,t) = \inf_{\mathbf{u}} \mathcal{J}(x,t;\mathbf{u}). \tag{4.5}$$

Here we consider a finite horizon problem for fixed T. In the infinite horizon problem with $T = \infty$ the pay-off function is changed to $u_0 \equiv 0$. Under mild conditions on f and c the following result can be proven (see [108, Section 1.3]). It establishes a fundamental connection between optimal control and first order differential equations.

Proposition 4.1.1. *For any $T > 0$, w is an element of $W^{1,\infty}(\mathbb{R}^n \times (0,T))$ and satisfies*

$$w_t(x,t) + \sup_{\mathbf{u}}\{b(x,\mathbf{u})\nabla w(x,t) + c(x,\mathbf{u})w(x,t) - f(x,\mathbf{u})\} = 0 \tag{4.6}$$

almost everywhere.

A similar result can be shown for the infinite horizon problem. Then a stationary Hamilton-Jacobi equation has to be considered. It turns out, that the generalized solutions of (4.6) satisfying (4.5) are viscosity solutions in the sense of Definition 4.1.2 (see [9]).

A further example for the natural occurrence of first order partial differential equations we present the motion of a front, i.e. the evolution of a $n-1$ dimensional hyper-surface in $\mathbb{R}^n$. For a given time t and a fixed point x on the front, the direction of evolution is defined as the normal direction in this point with respect to the front. The speed depends on the local curvature and on some given underlying flow. For a detailed discussion of the corresponding model and related real world problems we refer to [124] and the references therein. The basic technique is to introduce a level set function $\varphi(x,t)$ such that the solution curve

$$\varphi(x,t) = 0$$

describes the position of the front for all considered t. The authors have proven that this level set function satisfies the equation

$$\varphi_t(x,t) - F(K)|\nabla\varphi(x,t)| = 0. \tag{4.7}$$

F defines the speed of the evolution depending on the local curvature K. (4.7) is known as Eikonal equation and the existence of viscosity solutions for this problem is for example discussed in [108].

4.2 Variational Inequalities with First Order Hyperbolic Differential Operators

To the best of our knowledge, variational inequalities with first order differential operators were introduced in [14]. Whenever problems like (4.5) in addition contain state constraints as $y \in \mathbf{K}$ for some closed convex set $\mathbf{K} \subset \mathbb{R}^n$, the solution is no longer characterized by (4.6) but by a variational inequality. Moreover, the solutions of this inequalities are Lipschitz continuous for finite and infinite horizon problems, i.e. in the case of stationary and time dependent problems (see [14]). In [20] a similar result was achieved when considering dynamical systems for a given target region the state has to reach under consideration of obstacles for the solution trajectories, i.e. additional state constraints. A set $\mathbf{K}(t)$ is defined by the target region according the question whether the state starting from this point can be controlled into the target region or not. The resulting problem is defined as

$$\min\{y_t + H(x,\nabla y), y - \psi\} = 0 \text{ for all } x \in \mathbb{R}^n, \quad t > 0, \quad y(x,0) = u_0$$

and equivalent to an obstacle problem characterized by ψ (see Section 2.3).

Another field, where variational inequalities with first order differential operators occur naturally is given by differential games with stopping times. In [13] it has been shown that a first order VI attains a viscosity solution if it is the upper value of a certain model of a differential game with stopping time.

Besides classical generalized solutions, weak solutions are studied as well. In [118, 135] linear problems are considered for a given domain $\Omega \subset \mathbb{R}^n$. They are defined by the scalar operator

$$\mathbf{b} \cdot \nabla = \sum_{i=1}^{n} b_i(x) \frac{\partial}{\partial x_i}$$

for a given vector field $\mathbf{b} = (b_i(x))$ with

$$b_i \in C(\overline{\Omega}) \cap L^\infty(\Omega) \text{ for all } i = 1, ..., n \text{ and } \nabla \cdot \mathbf{b} \in L^\infty(\Omega).$$

It is further assumed that the vector field defines a smooth partition $\partial\Omega = \partial\Omega_+ \cup \partial\Omega_0 \cup \partial\Omega_-$ of the boundary in the sense that

$$\partial\Omega_+ = \{x \in \partial\Omega \backslash \Xi : \mathbf{b} \cdot \mathbf{n} > 0\}, \;\; \partial\Omega_- = \{x \in \partial\Omega \backslash \Xi : \mathbf{b} \cdot \mathbf{n} < 0\} \text{ and } \partial\Omega_0 = \partial\Omega \backslash (\partial\Omega_- \cup \partial\Omega_+).$$

Here $\mathbf{n}$ is the exterior normal of the boundary possibly not being defined on the set Ξ with zero $(n-1)$ surface measure. In addition to the advection term $\mathbf{b}$, the first order operator is defined by $b_0 \in L^\infty(\Omega)$ satisfying

$$b_0(x) - \frac{1}{2}\nabla \cdot b(x) \geq \underline{b} > 0$$

almost everywhere in Ω. This property, we will refer to as strong feasibility condition, was moreover used on [10]. The solutions of the VI's are elements of the problem depending Hilbert space

$$L^2_B(\Omega) = \{\varphi \in L^2(\Omega) : \mathbf{b} \cdot \nabla\varphi \in L^2(\Omega)\}$$

endowed with the graph norm (see Section 2.1.3). In addition, the subset

$$V_-(\Omega) = \overline{\{\varphi \in H^1(\Omega) : v = 0 \text{ a.e. in } \partial\Omega_-\}}^{L^2_B(\Omega)}$$

is considered. For given data $f \in L^2(\Omega)$ the vi is defined as

$$\text{find } y \in \tilde{\mathbf{K}}_- : \;\; (\mathbf{b} \cdot \nabla y + b_0 y - f, \varphi - y) \geq 0 \text{ for all } \varphi \in \tilde{\mathbf{K}}$$

for the closed convex sets $\tilde{\mathbf{K}} = \{\varphi \in L^2 : \varphi \geq 0 \text{ a.e.}\}$ and $\tilde{\mathbf{K}}_- = \tilde{\mathbf{K}} \cap V_-(\Omega)$. For further details we refer to [135]. In addition there will be a discussion of this solution concept in Chapter 6.

As a final example we introduce the variational model of sand pile growth (see [128]). Instead of classical obstacle problems as considered so far, this model has a different underlying closed convex set. Let Ω be a given domain where the sand will be piled. Assuming, that the corresponding boundary $\partial\Omega$ can be separated into $\partial\Omega_1$ and $\partial\Omega_2$, where the first part allows sand to flow out of the region and the second one can be interpreted as an impermeable wall, we define the Banach space $U = \{\varphi \in W^{1,4}(\Omega) | \varphi_{|\partial\Omega_1} = 0\}$ and the Bochner space

$$\mathcal{U} = L^4(0, T; U).$$

The growth of a sand pile with given intensity w of a distributed source letting the sand enter the system and a fixed terminal time T is described by the following variational inequality. Find $y \in \mathcal{K}$ with $y_t \in \mathcal{U}^*$ such that

$$\int_0^T \langle y_t(\tau) - w(\tau), y(\tau) - v(\tau)\rangle d\tau \geq 0 \text{ for all } v \in \mathcal{K}$$

with $\mathcal{K} := \{\varphi \in \mathcal{U} : |\nabla\varphi(\tau)| \leq \gamma \text{ for almost all } \tau \in (0, T)\}$. If a transport process is introduced in the domain, the operator in addition would have a first order part acting on the spatial variables. The constant $\gamma > 0$ depends on the properties of the sand and denotes the angle of repose of the material. If the material is piled on a domain that contains slopes exceeding the angle of repose, the process is no longer described by a variational but by a quasivariational inequality. In this case the feasible set $\mathcal{K}$ depends on the state y and we obtain

$$\text{find } y \in \mathcal{K}(y), \quad y_t \in \mathcal{U}^*, \quad \int_0^T \langle y_t(\tau) - w, y(\tau) - v(\tau) \rangle d\tau \geq 0 \text{ for all } v \in \mathcal{K}(y).$$

For details we refer to [128].

4.3 Optimal Control and Hyperbolic Partial Differential Equations

Concerning optimal control subject to Hamilton-Jacobi equations there are way less results available than in the case of second order PDE's. This mostly stems from the fact that adjoint calculus as in Section 3.2 is not available in general for such underlying equations. However, there are examples as [43] where the Eikonal equations has been controlled for a $\mathbb{R}^1$ domain.

4.3.1 Entropy Solutions

A comprehensive work concerning the optimal control of entropy solutions of scalar nonlinear hyperbolic conservation laws with source terms is [151]. Here the control is split into a distributed part acting in the source term and the initial condition. In this work the concept of shift derivatives was introduced to establish sensitivity and adjoint calculus for the corresponding optimization problems in one dimensional unbounded domains. Such shift derivatives provide a generalized variation of the solution operator of the nonlinear conservation law. The composition of this kind of derivatives for the underlying equation and suitable objective functionals is Fréchet differentiable and allows for the design of optimization algorithms. Extending this concept to systems is still an open problem (2×2 systems are for example considered in [24]) but for bounded domains some progress has been made (see e.g. [39]).

4.3.2 Viscosity Solutions

The optimal control of viscosity solutions for first-order Hamilton-Jacobi equations was considered in the context of mean field games. In the recent paper [60], the problem

$$\begin{aligned} &\min \int_0^T \int_{\mathbb{T}^n} K(t, x, u(x,t)) dx dt + \int_{\mathbb{T}^n} y(x, 0) dm_0 \\ &\text{s.t.} - y_t(x,t) + H(x, \nabla y(x,t)) = u(x,t) \\ &\quad y(T, x) = y_T(x) \end{aligned} \tag{4.8}$$

was investigated. Here m_0 is a finite measure, y is the viscosity solution of the underlying Hamilton-Jacobi equation and $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ is the n dimensional torus which in particular avoids the discussion of boundary conditions. In addition to providing the existence of minimizers and the introduction of the dual problem, the author characterizes minimizers of (4.8) by a system of certain partial differential equations related to mean field games. For further examples of optimal control problems subject to Hamilton-Jacobi equations we refer to [60] and the references therein.

4.4 Numerical Methods for Hyperbolic Partial Differential Equations

Given a first order equation, the method of characteristics allows the identification of curves called characteristics along which the differential equation becomes an ODE. Once this ordinary differential equation is found, it can be solved along the characteristic curves and transformed back into a solution for the original equation provided certain regularity assumptions are met. For details we refer to [52].

For hyperbolic conservation and balance laws a whole zoo of discretization methods is available known to converge to the correct entropy solution. An overview about available numerical methods for hyperbolic problems is presented in [102]. Besides first order accurate methods as the upwind method and Godunov schemes it also covers essentially non-oscillatory and weighted essentially non-oscillatory schemes, which are discussed in more detail in [145]. While for second order differential operator the accuracy of a method can be improved by using polynomials of high order to approximate the exact solution in the spatial variables, i.e. to use an increased number of nodal values for the computation of the approximation in each grid point, this does not carry over to first order equations, which are known to contain shocks. While high degree polynomials are approximating smooth parts of the solutions well, they fail at discontinuities since they tend to overshoot. WENO and ENO methods are designed to compensate this drawback by providing a method how to choose the nodes of the grid for the computation of the current unknown, the so called stencil. Thus they do not use a fixed stencil at all points of the discretization but vary it pointwise.

As shown in [146] and further discussed in [12], the solutions of a discretization scheme converge to the viscosity solution of a Hamilton-Jacobi equation, if the scheme is consistent, stable and monotone. Here, monotonicity has to be understood in the following sense.

Definition 4.4.1. *Let $u, v \in \mathbb{R}^n$ be given vectors and let $F : \mathbb{R}^n \to \mathbb{R}^n$ be a function realizing the discretization scheme.*
F is called monotone if

$$F(u) \geq F(v) \text{ whenever } u \geq v.$$

In context of single step methods for ordinary differential equations F be seen as the mapping from the current iterate to the next one. In the following example we construct a monotone discretization scheme for a particular first order partial differential equation which is a regularized version of the Eikonal equation.

Example 4.4. *Consider the Eikonal equation* (4.7)*. In this case the Hamiltonian is given as*

$$H(x, t, y, \nabla y) = |\nabla y| = \sqrt{y_{x_1}^2 + y_{x_2}^2}.$$

For a discretization, we introduce a uniform grid of width dx in a given spatial domain $\Omega = [x_{0_1}, x_1] \times [x_{0_2}, x_2] \subset \mathbb{R}^2$ which, for simplicity, is assumed to be rectangular. For the nodal values $x_{i,j} = (x_{0_1} + idx, x_{0_2} + jdx)$ and a fixed parameter t we define the nodal values of the solution y as

$$y_{i,j}^{(t)} = y(idx, jdx, t).$$

Moreover, we introduce the one-sided finite difference operators in directions x_1 *and* x_2

$$D^{\pm}_{x_1} y_{i,j} = y_{i\pm 1,j} - y_{i,j}, \quad D^{\pm}_{x_2} y_{i,j} = y_{i,j\pm 1} - y_{i,j}.$$

Two known possibilities for the discretization of the Euclidean norm are given as

$$|\nabla y_{i,j}| \approx \sqrt{dx^{-2} \max\{D^{-}_{x_1} y_{i,j}, D^{+}_{x_1} y_{i,j}, 0\}^2 + dx^{-2} \max\{D^{-}_{x_2} y_{i,j}, D^{+}_{x_2} y_{i,j}, 0\}^2} \tag{4.9}$$

and

$$\begin{aligned} |\nabla y_{i,j}| \approx \quad & \left[dx^{-2} \max\{D^{-}_{x_1} y_{i,j}, 0\}^2 + dx^{-2} \max\{D^{+}_{x_1} y_{i,j}, 0\}^2 \right. \\ & \left. + dx^{-2} \max\{D^{-}_{x_2} y_{i,j}, 0\}^2 + dx^{-2} \max\{D^{+}_{x_2} y_{i,j}, 0\}^2\right]^{1/2} \end{aligned} \tag{4.10}$$

The latter discretization scheme is more diffusive as pointed out in [142] and yields a differentiable spatial discretization. Both methods are so called Upwind discretizations.

Next we introduce the Hamilton-Jacobi equation

$$y_z - (1/\omega)((y_{x_1})^2 + (y_{x_2})^2 + \nu^2)^{1/2} - u = 0$$

for a strictly positive function $\omega : \mathcal{Q} \to \mathbb{R}, \omega \geq \underline{\omega} > 0$, *a small regularization parameter* $\nu \in \mathbb{R}_+$ *and an additional function* $u : \mathcal{Q} \to \mathbb{R}_+$. *This is a regularized version of the Eikonal equation with a function* ω *influencing the speed of the propagated front and a drift term* u. *In addition we consider a function* $\Phi : \partial\Omega \times [0, T]$ *with* $\Phi(x, t) = \Phi(t)$ *nondecreasing which defines a boundary condition. Utilizing the method of lines (see [66]) an discretizing the spatial variables by* (4.10), *we obtain the following semidiscrete system of ODE's for all* $x_{i,j}$ *with* $\{i, j\} \in I_0 = \{(i, j) : x_{i,j} \in \Omega \backslash \partial\Omega\}$.

$$\begin{aligned} \dot{\mathbf{y}}_{i,j}(z) \quad &= (1/\omega) \left[dx^{-2} \max\{D^{-}_{x_1} \mathbf{y}_{i,j}(z), 0\}^2 + dx^{-2} \max\{D^{+}_{x_1} \mathbf{y}_{i,j}(z), 0\}^2 \right. \\ & \left. + dx^{-2} \max\{D^{-}_{x_2} \mathbf{y}_{i,j}(z), 0\}^2 + dx^{-2} \max\{D^{+}_{x_2} \mathbf{y}_{i,j}(z), 0\}^2 + \nu\right]^{1/2} + \mathbf{u}_{i,j}(z) \end{aligned} \tag{4.11}$$

Whenever $x_{i\pm 1,j}$ *or* $x_{i,j\pm 1}$ *is located on* $\partial\Omega$, *the corresponding* $D^{\pm}_{x_l} \mathbf{y}_{i,j}(z)$ *has to be replaced by the difference* $\Psi(z) - \mathbf{y}_{i,j}(z)$ *for* $l = 1, 2$.

The discretization scheme (4.10) *has been chosen as the resulting system of ordinary differential equations is differentiable with respect to* $\mathbf{y}$ *which is important if we want to consider problems of optimal control. Utilizing the explicit Euler method for the time stepping in* (4.11) *we obtain the fully discretized System*

$$\begin{aligned} \mathbf{y}^{k+1}_{i,j} \quad &= dt[(1/\omega^k_{i,j})(dx^{-2} \max\{D^{-}_{x_1} \mathbf{y}^k_{i,j}, 0\}^2 + dx^{-2} \max\{D^{+}_{x_1} \mathbf{y}^k_{i,j}, 0\}^2 \\ & \quad + dx^{-2} \max\{D^{-}_{x_2} \mathbf{y}^k_{i,j}, 0\}^2 + dx^{-2} \max\{D^{+}_{x_2} \mathbf{y}^k_{i,j}, 0\}^2 + \nu)^{1/2} + \mathbf{u}^k_{i,j}] + \mathbf{y}^k_{i,j} \\ &= dt[1/(\omega^k_{i,j} dx)(\max\{D^{-}_{x_1} \mathbf{y}^k_{i,j}, 0\}^2 + \max\{D^{+}_{x_1} \mathbf{y}^k_{i,j}, 0\}^2 \\ & \quad + \max\{D^{-}_{x_2} \mathbf{y}^k_{i,j}, 0\}^2 + \max\{D^{+}_{x_2} \mathbf{y}^k_{i,j}, 0\}^2 + \nu dx^2)^{1/2} + \mathbf{u}^k_{i,j}] + \mathbf{y}^k_{i,j} \end{aligned} \tag{4.12}$$

To keep the notation short we introduce the vector

$$\Gamma(\mathbf{y}_{i,j}) = (\max\{D^{-}_{x_1} \mathbf{y}_{i,j}, 0\}, \max\{D^{+}_{x_1} \mathbf{y}_{i,j}, 0\}, \max\{D^{-}_{x_2} \mathbf{y}_{i,j}, 0\}, \max\{D^{+}_{x_2} \mathbf{y}_{i,j}, 0\})^{\top}.$$

and the mapping

$$\begin{aligned} F(\mathbf{y}_{i,j}) = \quad & (\max\{D^-_{x_1}\mathbf{y}_{i,j},0\}^2 + \max\{D^+_{x_1}\mathbf{y}_{i,j},0\}^2 \\ & + \max\{D^-_{x_2}\mathbf{y}_{i,j},0\}^2 + \max\{D^+_{x_2}\mathbf{y}_{i,j},0\}^2 + (\nu dx)^2)^{1/2} \\ = \quad & (\Gamma(\mathbf{y}_{i,j})\cdot\Gamma(\mathbf{y}_{i,j}) + (\nu dx)^2)^{1/2} \end{aligned}$$

with $\cdot$ *denoting the standard scalar product of* $\mathbb{R}^4$*.*

Proposition 4.4.1. *Consider* (4.12)*. Then the scheme is*

a) monotone for $dz \leq \underline{\omega} dx/2$*,*

b) consistent,

c) stable.

Proof. a) For the claimed monotonicity, we have to prove, that $\tilde{\mathbf{y}}^{k+1}_{i,j} \geq \mathbf{y}^{k+1}_{i,j}$ holds, whenever $\tilde{\mathbf{y}}^k_{i,j} \geq \mathbf{y}^k_{i,j}$ is satisfied element wise. For $i,j \in I^0$ (4.12) provides

$$\tilde{\mathbf{y}}^{k+1}_{i,j} - \mathbf{y}^{k+1}_{i,j} = dz/(\omega dx)[F(\tilde{\mathbf{y}}^k_{i,j}) - F(\mathbf{y}^k_{i,j})] + \tilde{\mathbf{y}}^k_{i,j} - \mathbf{y}^k_{i,j} \tag{4.13}$$

By the fundamental Theorem of calculus we obtain for

$$D(\tilde{\mathbf{y}}^k_{i,j} - \mathbf{y}^k_{i,j}) = (D^-_{x_1}(\tilde{\mathbf{y}}^k_{i,j} - \mathbf{y}^k_{i,j}), D^+_{x_1}(\tilde{\mathbf{y}}^k_{i,j} - \mathbf{y}^k_{i,j}), D^-_{x_2}(\tilde{\mathbf{y}}^k_{i,j} - \mathbf{y}^k_{i,j}) D^+_{x_2}(\tilde{\mathbf{y}}^k_{i,j} - \mathbf{y}^k_{i,j}))^\top$$

the following estimate.

$$\begin{aligned} F(\tilde{\mathbf{y}}^k_{i,j}) - F(\mathbf{y}^k_{i,j}) &= \int_0^1 \frac{d}{dl} F(\mathbf{y}^k_{i,j} + l(\tilde{\mathbf{y}}^k_{i,j} - \mathbf{y}^k_{i,j}) dl \\ &= \int_0^1 \frac{\Gamma(\mathbf{y}^k_{i,j} + l(\tilde{\mathbf{y}}^k_{i,j} - \mathbf{y}^k_{i,j}) \cdot D(\tilde{\mathbf{y}}^k_{i,j} - \mathbf{y}^k_{i,j})}{F(\mathbf{y}^k_{i,j} + l(\tilde{\mathbf{y}}^k_{i,j} - \mathbf{y}^k_{i,j}))} dl \\ &\geq \int_0^1 \frac{\Gamma(\mathbf{y}^k_{i,j} + l(\tilde{\mathbf{y}}^k_{i,j} - \mathbf{y}^k_{i,j})(\mathbf{y}^k_{i,j} - \tilde{\mathbf{y}}^k_{i,j})}{F(\mathbf{y}^k_{i,j} + l(\tilde{\mathbf{y}}^k_{i,j} - \mathbf{y}^k_{i,j}))} dl \end{aligned}$$

where the inequality comes from the assumption $\tilde{\mathbf{y}}^k_{i,j} \geq \mathbf{y}^k_{i,j}$ implying $(\mathbf{y}^k_{i,j} - \tilde{\mathbf{y}}^k_{i,j}) \leq 0$. By the equivalence of norms on $\mathbb{R}^n$ we obtain

$$\Gamma(\mathbf{y}^k_{i,j} + l(\tilde{\mathbf{y}}^k_{i,j} - \mathbf{y}^k_{i,j}) = |\Gamma(\mathbf{y}^k_{i,j} + l(\tilde{\mathbf{y}}^k_{i,j} - \mathbf{y}^k_{i,j})|_{l^1} \geq (1/2)|\Gamma(\mathbf{y}^k_{i,j} + l(\tilde{\mathbf{y}}^k_{i,j} - \mathbf{y}^k_{i,j})|_{l^2}$$

and since $F(\mathbf{y}^k_{i,j} + l(\tilde{\mathbf{y}}^k_{i,j} - \mathbf{y}^k_{i,j}) \geq |\Gamma(\mathbf{y}^k_{i,j} + l(\tilde{\mathbf{y}}^k_{i,j} - \mathbf{y}^k_{i,j}))|_{l^2}$ by construction we can estimate

$$\tilde{\mathbf{y}}^{k+1}_{i,j} - \mathbf{y}^{k+1}_{i,j} \geq (\tilde{\mathbf{y}}^k_{i,j} - \mathbf{y}^k_{i,j})/2.$$

Consequently (4.13) can be estimated as

$$\tilde{\mathbf{y}}^{k+1}_{i,j} - \mathbf{y}^{k+1}_{i,j} \geq (1 - dz/(2\omega dx))(\tilde{\mathbf{y}}^k_{i,j} - \mathbf{y}^k_{i,j})$$

which has to be nonegative for monotonicity of the scheme and satisfied for

$$dz < 2\underline{\omega} dx$$

The case of $i, j \notin I^0$ can be shown analogously.

b) The proof follows the line of [122] where a similar assertion has been proven. Given a sufficiently smooth function $\phi : \mathbb{R}^3 \to \mathbb{R}$ we employ the Taylor expansion at an arbitrary point $q \in \mathcal{Q}$,
$$\phi(q+h) = \phi(q) + \nabla\phi(q) \cdot h + O(h^2)$$
and obtain at any $x = x_{i,j}$
$$\begin{aligned}
& |(\phi_{x_1}(x)^2 + \phi_{x_2}(x)^2 + \nu^2)^{1/2} - \\
& (\max\{\tfrac{\phi(x+dx_1)-\phi(x)}{dx_1}, 0\}^2 + \max\{\tfrac{\phi(x-dx_1)-\phi(x)}{dx_1}, 0\}^2 + \max\{\tfrac{\phi(x+dx_2)-\phi(x)}{dx_2}, 0\}^2 \\
& \qquad + \max\{\tfrac{\phi(x-dx_2)-\phi(x)}{dx_2}, 0\}^2 + \nu^2)^{1/2}| \\
= \; & |(\phi_{x_1}(x)^2 + \phi_{x_2}(x)^2 + \nu^2)^{1/2} - \\
& (\max\{\phi_{x_1}(x) + O(dx_1), 0\}^2 + \max\{-\phi_{x_1}(x) + O(dx_1), 0\}^2 \\
& \qquad + \max\{\phi_{x_2}(x) + O(dx_2), 0\}^2 + \max\{\phi_{x_2}(x) + O(dx_2), 0\}^2 + \nu^2)^{1/2}| \\
= \; & |(\phi_{x_1}(x)^2 + \phi_{x_2}(x)^2 + \nu^2)^{1/2} - \\
& (\max\{\phi_{x_1}(x), 0\}^2 + \max\{-\phi_{x_1}(x), 0\}^2 \\
& \qquad + \max\{\phi_{x_2}(x), 0\}^2 + \max\{\phi_{x_2}(x), 0\}^2 + O(dx) + \nu^2)^{1/2}| \\
= \; & |(\phi_{x_1}(x)^2 + \phi_{x_2}(x)^2 + \nu^2)^{1/2} - (\phi_{x_1}(x)^2 + \phi_{x_2}(x)^2 + O(dx) + \nu^2)^{1/2}| \\
\leq \; & (1/\nu)|O(dx)| = O(dx)
\end{aligned}$$
Now the claimed consistency follows from the consistency of the explicit Euler method with the time derivative.

c) Due to the occuring max operator the discretization scheme is even *degenerate elliptic* in the sense of [123]. Given an abstract scheme, degenerate ellipticity is stronger than monotonicity and implies that a discrete comparison principle holds for the equation (see [122]). From this property stability follows directly as shown in the presented reference.
□

In the light of the comments above Definition 4.4.1 the last result ensures, that solutions to the discretized problems converge to the viscosity solution of (4.10) *if the step size w.r.t. t is chosen properly.*

The suggested discretization scheme can be used for the problem in Chapter 5 yielding a discretization of the occuring forward problem.
There are several ways to define monotone discretization schemes. According to [123] certain elementary operations and functions preserve monotonicity of a discretization scheme. Thus, starting with a monotone scheme for most the basic elements of a given hyperbolic partial differential equation, a monotone difference scheme can be constructed if the elementary functions generating the PDE from the chosen basic elements are monotonicity preserving and applied to the discretization.
This method of construction is similar to algorithmic differentiation (see [64]) for the automatic generation of directional derivatives of functions.
Note that recently much progress has been done in the context of solving systems of ODE's where the right hand side is just Lipschitz continuous and the nondifferentiabilities only stem from the absolute value function (and consequently max and min). In [61] a piecewise linearization of the right hand side is introduced and it is shown, that the generalized midpoint rule based on this linearization maintains a global convergence order of 2 as in the differentiable

case. In [99] this integration method was applied to problems of data assimilation. They, for example, considered the $1 - D$ shallow water equation with a second order accurate discretization scheme for the spatial variables. The particular choice provided a Lipschitz continuous right hand side since the second order accuracy of the scheme was achieve by a so called flux limiter approach with minmod limiter (see [102]) consisting of certain max and min operations. We have compared several instances of the forward problems (4.11) numerically for the discretizations (4.9) and (4.10) to identify the claimed increased diffusion of the latter scheme. Although the nondifferentiable discretization performed slightly better in the test runs, the increased effort of changing from an explicit Euler time stepping to an the implicit generalized midpoint rule did not pay off. In addition, monotonicity of a discretization scheme with implicit time stepping is harder to establish. Further numerical experiments on the generalized midpoint rule can for example be found in [19].
A further class of monotone schemes are the fast marching methods as discussed in [143] and fast sweeping methods as presented in [130, 150]. They are especially suited to Eikonal like problems and will be used later in this thesis.

5 Open Pit Mine Planning

As described in Chapter 1, the usual mathematical procedure for solving the open pit mine planning problem is to separate the ore body into a large number of blocks. Then an optimal excavation sequence of these blocks is identified by a integer- or mixed-integer program obtained from this discretization. This is solved by the comprehensive methodology available for this class of problems.

In this section we analyze the problem of optimal open pit mine planing in a function space setting. The modeling with continuous functions was presented in [4] and further discussed in [62].

A first similar approach was introduced in 1975 in [113, 114] but not further developed. Moreover a further method has been elaborated in [47]. In the function space model the relation guaranteeing the physical stability of the open pit has to be defined, instead of a block wise manner as in mathematical problems arising from the block model, on a point wise level.

The stability condition is strongly nonconvex. We will present a reformulation of the model based on continuous functions where the constraint is convex. Moreover, the resulting problem is a problem of optimal control governed by a nonlinear first order PDE.

This chapter is organized as follows. In Section 5.1 we recall the model for open pit mine planning utilizing continuous functions introduced in [4] and present selected results. In Section 5.2 we discuss the reformulation of the time dependent problem that involves a first order hyperbolic PDE of Eikonal type and corresponding viscosity solution. Utilizing the results concerning this objects we introduce the time dependent or dynamic open pit mine planning problem as a problem of optimal control of viscosity solutions to a Hamilton-Jacobi PDE. In Section 5.3 we define the problem and prove the existence of a solution. In addition we introduce auxiliary problems based on a regularization of the first order differential operator, prove the existence of solutions and finally present a convergence result for solutions of these problems. We close with a brief numerical experiment in Section 5.4 where a particular profile, the so called ultimate pit, is obtained.

5.1 Model Based in Continuous Functions

Let $\Omega \subset \mathbb{R}^n$ be a bounded, connected domain with sufficiently smooth boundary $\partial\Omega$ for $n = 1, 2$.

Definition 5.1.1. *The state of excavation of the open pit mine at any particular time is defined by a function $p \in C(\bar{\Omega})$, called* ***profile****, satisfying the following conditions.*

$$p(x) - p_0(x) \geq \; 0 \textit{ for } x \in \overline{\Omega} \tag{5.1a}$$
$$p(x) - p_0(x) = \; 0 \textit{ for } x \in \partial\Omega \tag{5.1b}$$
$$\limsup_{\tilde{x}\to x \leftarrow \hat{x}} \frac{|p(\hat{x}) - p(\tilde{x})|}{|\hat{x} - \tilde{x}|} \leq \; \omega(x, p(x)) \textit{ for } x \in \overline{\Omega} \tag{5.1c}$$

The value of $z = p(x)$ for $x \in \overline{\Omega}$ represents the depth of the mine subject to a certain reference level. Figure 5.1 depicts a one dimensional example for such profile functions.

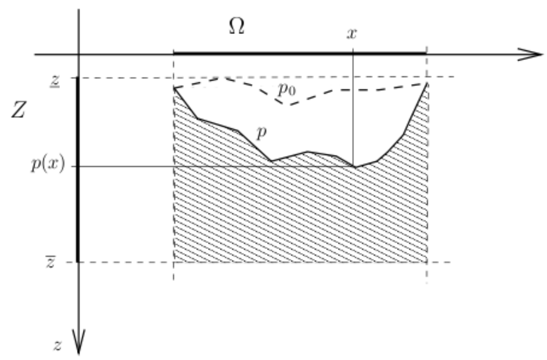


Figure 5.1: Example of a continuous profile function for a one dimensional domain

5.1a ensures, that any profile has to be below the initial profile p_0. Thus no piling of material is allowed. 5.1b ensures, that the mine is continuously connected to the surroundings. Finally, 5.1c represents the so called **pointwise stability condition**. The quotient induces, whenever it is finite, a sharp local Lipschitz constant for p around x. The function ω defined in the volume $\overline{\Omega} \times Z$ is an upper bound on this limiting local Lipschitz constant. It defines the maximal local slope such that p is stable and may vary in $\overline{\Omega} \times Z$ depending on the local geotechnical properties of the ore body. We assume the following elementary property.

$$0 < \underline{\omega} \leq \omega(x, z) \text{ for all } (x, z) \in \overline{\Omega} \times Z \tag{5.2}$$

Here, $\underline{\omega}$ is a small positive constant reflecting the fact, that any material except liquids allows for at least small angles of repose. The set of admissible profiles is defined as

$$\mathcal{P} := \{p \in C(\overline{\Omega}) \mid p \text{ satisfies 5.1a, 5.1b and 5.1c}\}.$$

For further discussions on the function ω we refer to [4]. For all results in this section, upper semicontinuity of ω is sufficient while in the further parts of the chapter we assume more regularity. Since all admissible profiles are Lipschitz continuous with a uniform constant, they are not only bounded from below but also from above by a value $\overline{z} > \underline{z}$ due to physical and operational conditions. Thus for any admissible profile we assume $p(x) \in Z \equiv [\underline{z}, \overline{z}]$ for $x \in \overline{\Omega}$. Without loss of generality we may assume $\underline{z} \geq 0$. Since all feasible profile functions are in particular locally Lipschitz continuous, they admit a derivative almost everywhere by Rademacher's Theorem A.1.2. At all points x, where p is differentiable, the stability constraint 5.1c reduces to $|\nabla p(x)| \leq \omega(x, p(x))$ providing, that 5.1c is satisfied if

$$|\nabla p(x)| \leq \omega(x, p(x)) \text{ holds for a.e. } x \in \Omega. \tag{5.3}$$

The proof of the following claims can be found in [4].

Proposition 5.1.1. *If ω is upper semicontinuous, $\mathcal{P}$ is compact and has an empty interior in $C(\overline{\Omega})$.*

Note, that the result is more than a straight forward application of Lemma A.1.1 since also the local Lipschitz constant is preserved for limits of feasible functions. Consequently any functional $F : C(\overline{\Omega}) \to \mathbb{R}$ which is continuous with respect to $\|\cdot\|_{L^\infty(\Omega)}$ attains a minimum and a maximum on $\mathcal{P}$ by the Weierstrass Theorem. This applies in particular to the distances $F(p) \equiv \|p - \tilde{p}\|_{L^\infty(\Omega)}$ for any fixed $\tilde{p} \in L^\infty(\Omega) \supset C(\overline{\Omega}) \supset \mathcal{P}$. Hence we have (non unique) least distance projections from $L^\infty(\Omega)$ to $\mathcal{P}$.

Proposition 5.1.2. *If ω is upper semicontinuous, $\mathcal{P}$ is closed with respect to pointwise minima and maxima in the following sense. For any subset $\tilde{\mathcal{P}} \subset \mathcal{P}$, the functions $\underline{p}$ and $\bar{p}$ defined by*

$$\underline{p}(x) \equiv \inf\{p(x) | p \in \tilde{\mathcal{P}}\} \quad \textit{and} \quad \overline{p}(x) \equiv \sup\{p(x) | p \in \tilde{\mathcal{P}}\}$$

also belong to $\mathcal{P}$. Consequently, $\mathcal{P}$ contains a unique maximal element $\overline{p}_u \equiv \max\limits_{p \in \mathcal{P}}\{p\}$.

For further properties of $\mathcal{P}$ we refer to [4]. In addition to ω the modeling of the open pit problem relies on two further real valued functions, namely the excavation density $e \in L^\infty(\Omega \times Z)$ and the gain density $g \in L^\infty(\Omega \times Z)$ measuring the effort to remove material from the ore body and the gain which is realized if the material is sold. Thus they are the counterpart of the corresponding weight in the block model discussed in Chapter 1. The effort constraint in addition fulfills

$$e(x, z) \geq e_0 > 0 \text{ for } (x, z) \in \Omega \times Z.$$

e and g are allowed to have jumps due transitions between different types of material in the ore body. For any two given profiles $p_1 \leq p_2$ the integral

$$E([p_1, p_2]) \;\equiv\; \int_\Omega \int_{p_1(x)}^{p_2(x)} e(x, z) \mathrm{d}z \mathrm{d}x$$

represents the **effort** to excavate the material between profile p_1 and p_2. The integral

$$G([p_1, p_2]) \;\equiv\; \int_\Omega \int_{p_1(x)}^{p_2(x)} g(x, z) \mathrm{d}z \mathrm{d}x$$

represents the total value or **gain** of the material between p_1 and p_2 (without considering a discount rate). Notice that the function $g(x, z)$ may take negative values while e has to be strictly positive reflecting the fact, that gaining depth is only possible, if a certain effort is invested. For $p_1 = p_0$ we abbreviate $G(p) \equiv G([p_0, p])$ and $E(p) \equiv E([p_0, p])$.

Proposition 5.1.3. *$E(p)$ and $G(p)$ are Lipschitz continuous on $C(\overline{\Omega})$ with constants $\|e\|_\infty|\Omega|$ and $\|g\|_\infty|\Omega|$ respectively.*

The optimization problems of open pit mine planning without considering time are given as follows.

$$\begin{aligned} \max \quad & G(p) \\ \text{s.t.} \quad & p \in \mathcal{P} \end{aligned} \tag{FOP}$$

Here we try to identify the profiles generating the maximal gain. In addition we introduce the capacitated final open pit problem

$$\begin{aligned} \max \quad & G(p) \\ \text{s.t.} \quad & p \in \mathcal{P} \\ & E(p) \leq \overline{E} \end{aligned} \tag{CFOP}$$

where the profiles generating the maximal revenue of the mine are identified among those satisfying an effort constraint. According to Proposition 5.1.3, the remarks below and Proposition 5.1.2 both problems admit a solution. Moreover, the solution set of (FOP) has an interesting structure as discussed in [4].
Concerning optimal sequences of profiles, the model introduced in the paper considers paths

$$P : [0,T] \mapsto \mathcal{P}$$

that are monotonic, i.e. $t_1, t_2 \in [0,T]$ with $t_1 \leq t_2$ imply for $p_1 = P(t_1)$ and $p_2 = P(t_2)$ that $p_1(x) \leq p_2(x)$ for all $x \in \Omega$. Naturally, the function $E(P(t))$ must be monotonically increasing as well. We assume that there exists an absolutely continuous function $\mathcal{K} : [0,T] \to \mathbb{R}^+$ with

$$\mathcal{K}(t) \ \equiv \ \int_0^t \kappa(\tau)\mathrm{d}\tau$$

representing the mining capacity in the time interval $[0,t]$, with density $\kappa \in L^\infty(0,T), \kappa \geq \underline{\kappa} > 0$. Finally, we impose the capacity condition on P

$$E([P(t_1), P(t_2)]) = E(P(t_2)) - E(P(t_1)) \leq \mathcal{K}(t_2) - \mathcal{K}(t_1) = \int_{t_1}^{t_2} \kappa(\tau)\mathrm{d}\tau \ \text{ for } t_1 \leq t_2 \tag{5.4}$$

and introduce the set of feasible excavation paths

$$\begin{aligned} \mathcal{U} =\{ & P \in C([0,T];\mathcal{P}) \mid \\ & p_0 \leq P(t_1) \leq P(t_2), \ \ E([P(t_1), P(t_2)]) \leq \mathcal{K}(t_2) - \mathcal{K}(t_1) \ \text{ for } 0 \leq t_1 \leq t_2 \leq T\}. \end{aligned}$$

Proposition 5.1.4. *All paths $P \in \mathcal{U}$ satisfying the capacity condition fulfill*

$$\|P(t_1) - P(t_2)\|_\infty \leq \left[\frac{\|\kappa\|_\infty}{e_0\pi} + 2\overline{\omega}\right](t_1 - t_2)^{1/3}.$$

Consequently, the elements of $\mathcal{U}$ are Hölder equicontinuous and the feasible set is compact in $C([0,T]; C(\overline{\Omega}))$ which is identified with $C([0,T] \times \overline{\Omega})$ endowed with the uniform norm $\|\cdot\|_\infty$.

Let $\varphi \in C^1(0,T)$ be a monotonically decreasing discount function with $\varphi(0) = 1$ and $\varphi(T) < 1$ for some fixed time period $[0,T]$. For paths $P(x,t) \equiv P(t)(x)$ that are smooth in time we define the net present value as

$$\hat{G}(P) \equiv \int_0^T \varphi(t) \int_\Omega g(x, P(x,t))\, \mathrm{d}x\, dP(t) = \int_\Omega \int_0^T \varphi(t) g(x, P(x,t)) P_t(x,t) \mathrm{d}t \mathrm{d}x$$

suggesting that $P(x,t)$ has to be differentiable with respect to t. Integration by parts yields a formulation without this requirement.

$$\begin{aligned}\hat{G}(P) &= \left[\varphi(t) \int_\Omega \int_{P(0,x)}^{P(x,t)} g(x,z) \mathrm{d}z \mathrm{d}x\right]_0^T + \int_\Omega \int_0^T [-\varphi'(t)] \int_{P(0,x)}^{P(x,t)} g(x,z) \mathrm{d}z \mathrm{d}t \mathrm{d}x \\ &= \varphi(T) \left(\int_\Omega \hat{g}(x, P(T,x)) - \hat{g}(x, P(0,x)) \mathrm{d}x\right) \\ &\quad + \int_\Omega \int_0^T [-\varphi'(t)]\, [\hat{g}(x, P(x,t)) - \hat{g}(x, P(0,x))]\, \mathrm{d}t \mathrm{d}x \end{aligned} \tag{5.5}$$

Here $\hat{g}(x,z) \equiv \int_{p_0(x)}^z g(x,\tau) d\tau$ is the antiderivative based on the initial profile. The usual choice for the discount function is $\varphi(t) = e^{-rt}$ with some fixed discount rate $r > 0$. For feasible excavation paths the first term in (5.5) represents the value of the total excavated material of the path discounted by $\varphi(T)$. The second term represents a correction of this value due to the variation of φ (with $-\varphi'(t) > 0$). Note that $\mathcal{U}$ not includes a particular initial state $P(0,\cdot) = p_0(\cdot)$. Thus, the optimization problem in the so called dynamic trajectory planning case, the Capacitated Dynamic Open Pit Problem, is the following one.

$$\begin{array}{rl} \max & \hat{G}(P) \\ \text{s.t.} & P \in \mathcal{U} \\ & P(0) = p_0 \end{array} \tag{CDOP}$$

Proposition 5.1.5. *For arbitrary paths $P, Q \in \mathcal{U}$ we have*

$$\left|\hat{G}(P) - \hat{G}(Q)\right| \leq 2|\Omega|\, \|g\|_\infty\, \|P - Q\|_\infty$$

Thus the objective is continuous and, by the Weierstrass Theorem, (CDOP) attains a solution.

5.2 Reformulation the Problem

5.2.1 The Stability Condition as a Partial Differential Equation

The main advantage of the model based on continuous functions introduced in the preceding section is the flexible imposition of the stability constraint (5.3). This constraint is not convex and carries several other problems in the theoretical considerations (see [62]). Thus we present a new approach to deal with the stability constraint. It is based on the following change of variables. While, for a fixed position $x \in \Omega$, an excavation path $\mathcal{P}$ in the sense of Definition 5.1.1 assigns a certain point in time to the corresponding depth, the object we consider assigns, roughly speaking, the time of excavation to each coordinate in the volume and therefore be refereed to as *time labeling function*. Figure 5.2 sketches the change of variables and the corresponding qualitative behavior of excavation paths and time labeling functions for fixed $x \in \Omega$.

Obviously, discontinuities are introduced with respect to depth z. This is a consequence of

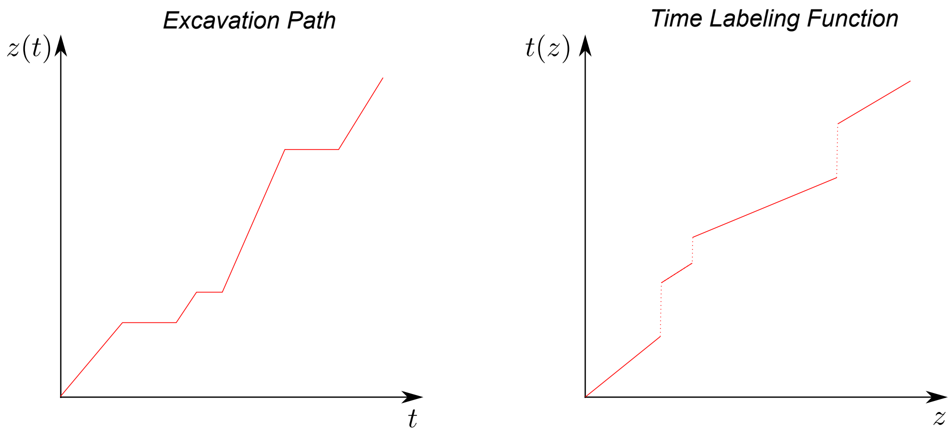


Figure 5.2: Qualitative behavior of Excavation Paths and time labeling functions

the observation, that the excavation process at certain parts of the mine might be stopped since excavating other parts of the mine yields material resulting in a higher total gain of the mining operation under consideration of the discount rate in the objective. Here we discuss possibilities to model such functions.

First, the discontinuities in Figure 5.2 could be modeled by set valued functions which assign the actual time of excavation to each coordinate, where this quantity is unique and the whole time interval between stopping and starting again to points, where the excavation process pauses for some reason. Although there exist a wide theory on set valued analysis and the related calculus (see, e.g. [5]) and also some numerical tools for this rather special case of set or interval valued functions (see, e.g. [3]), we decided against the usage of such models.

A second possible approach would be the consideration of lower semicontinuous functions. Unfortunately, it is not possible to model the Open Pit Mine Planning problem with these functions alone as they do not form a vector- let alone a Banach space. Thus the only possibility for the usage of this class of functions would be to take a larger function space containing at least a significant amount of lower semicontinuous functions and using the fact that all lower semicontinuous functions form a pointed cone as one can easily verify. As function space we suggest the space of functions of bounded variation, $\mathbf{BV}(\mathcal{Q})$. This approach results in a mathematical mathematical program with a cone constraint as considered in [67] but is not further elaborated in this thesis. Note that the cone of lower semicontinuous functions is not described by some pointwise condition.

A third possibility of considering time labeling functions is, to restrict them to a class of functions without jumps and enough regularity to quantify the growth behavior with respect to depth. Since this is our choice of working with the change of variables, we make these objects more precise at this point. Note that we have decided for this possibility since it provides a problem of optimal control in the end.

Definition 5.2.1. *A Lipschitz continuous function*

$$\mathcal{T} : \mathbb{R}^3 \to \mathbb{R}^+$$

that is strictly monotonically increasing with respect to the vertical coordinate z is called ***time labeling function (TLF)****.*

Note that the Lipschitz continuity and strict monotonicity implies

$$\mathcal{T}_z(x,z) > 0$$

almost everywhere. The requirement of strict positivity results from the natural physical observation, that a gain of depth can only be realized by investing time for the excavation process. For the remainder we strengthen the assumption to

$$\mathcal{T}_z(x,z) \geq \nu > 0$$

wherever the partial derivative is defined at all for $0 < \nu \ll 1$ sufficiently small. The derivative with respect to z can be interpreted as measure how much a process tries to pause at a certain point in the volume. The assumed regularity provides the possibility, to recover profile functions from a given TLF according to the following result.

Lemma 5.2.1. *For any $t \geq 0$ there exist a unique locally Lipschitz continuous function $p_t : \Omega \to \mathbb{R}^+$ satisfying*

$$\mathcal{T}(x, p_t(x)) = t. \tag{5.6}$$

Proof. As a consequence of the mean value Theorem, for all $t \in range(\mathcal{T})$ and for all $x \in \Omega$, there exist a set $\Theta_t(x)$ such that

$$\mathcal{T}(x,z) = t \quad \forall z \in \Theta_t(x).$$

The assumed strict monotonicity of $\mathcal{T}$ with respect to the vertical coordinate ensures, that $\Theta_t(x)$ is a singleton for all feasible pairings (x,t). Thus, the function $p_t : \Omega \to \mathbb{R}^+$ is well defined for any given t. Now fix an arbitrary $x_0 \in \Omega$ and the corresponding $p_t(x_0)$. The generalized implicit function Theorem [38, p 256] implies the existence of a neighborhood $x_0 \in \mathcal{O} \subset \Omega$ and a Lipschitz continuous function $\eta : \mathcal{O} \to \mathbb{R}$ with $\eta(x_0) = p_t(x_0)$ such that for every $x \in O$ we have $\mathcal{T}(x, \eta(x)) = t$. As p_t is uniquely defined for every $x \in \Omega$, the equality $p_t = \eta$ has to hold in O. Since x_0 was chosen arbitrary, p_t has to be locally Lipschitz continuous on Ω. □

Let $\mathcal{T}$ be given, fix a time t and consider (5.6) for p_t. Due to Lemma 5.2.1 we can differentiate the expression with respect to x_1, x_2 almost everywhere. As for the Implicit Function Theorem we obtain by the chain rule

$$\frac{d}{dx_i}\mathcal{T}(x,p_t(x)) = \underbrace{\frac{\partial}{\partial x_i}\mathcal{T}(x,p_t(x))}_{s_i} + \frac{\partial}{\partial z}\mathcal{T}(x,p_t(x))\frac{d}{dx_i}p_t(x) = 0.$$

Rearranging the terms and taking norms we further get

$$\sqrt{s_1^2 + s_2^2} = \mathcal{T}_z(x,p_t(x))|\nabla p_t(x)| \leq \mathcal{T}_z(x,p_t(x))\omega(x,p_t(x)).$$

The inequality has to be satisfied if the pointwise geotechnical stability condition (5.3) holds for the recovered profile. Generalizing the observation to the whole domain we call a TLF

physically stable if

$$|\nabla \mathcal{T}(x,z)| \leq \mathcal{T}_z(x,z)\omega(x,z) \tag{5.7}$$

holds for almost every $(x,z) \in \mathcal{Q}, z \in [0,\bar{z}]$ and $\bar{z}$ to be determined. Here the gradient is used as in Chapter 4 and represents the partial derivatives of $\mathcal{T}$ with respect to the spatial variables only. This definition will be used for the remainder of the chapter. Introducing a non negative slack function $u : \mathcal{Q} \to \mathbb{R}^+$ we reformulate (5.7) and obtain

$$\mathcal{T}_z(x,z) - \frac{1}{\omega(x,z)}|\nabla \mathcal{T}(x,z)| = u(x,z) \tag{5.8}$$

which is satisfied pointwise almost everywhere in $\mathcal{Q}$ for stable TLF with corresponding u. By positivity of u, monotonic growth of the TLF with respect to z is already ensured but strict monotonical increasing behavior might still be violated since $|\nabla \mathcal{T}(x,z)| = u(x,z) = 0$ can hold simultaneously. To overcome this issue we will consider, instead of (5.8), the equation

$$\mathcal{T}_z(x,z) - \frac{1}{\omega(x,z)}|\nabla \mathcal{T}(x,z)|_{[\nu]} = u(x,z). \tag{PDE^0}$$

with the smoothed Euclidean norm, $|q|_{[\nu]} = (q_1^2 + ... + q_n^2 + \nu^2)^{1/2}$ with $q \in \mathbb{R}^n$, for a parameter $0 < \nu \ll 1$. Note the similarity of the resulting underlying partial differential equation to the propagation of fronts as introduced in Chapter 4 since profiles might be interpreted as the front between excavated and still to be excavated material. As a useful side effect, the nonlinear term becomes differentiable by this regularization although introduced for other reasons. Equation (PDE^0) strengthens the stability condition since we find

$$0 \leq \mathcal{T}_z - \frac{1}{\omega}|\nabla \mathcal{T}|_{[\nu]} \leq \mathcal{T}_z - \frac{1}{\omega}|\nabla \mathcal{T}|$$

by nonegativity of ω and $|\nabla \mathcal{T}(x,z)|_{[\nu]} > |\nabla \mathcal{T}(x,z)|$. For physically stable profiles we have the following result.

Lemma 5.2.2. *The set of Lipschitz continuous TLF satisfying*

$$|\nabla \mathcal{T}(x,z)|_{[\nu]} \leq \mathcal{T}_z(x,z)\omega(x,z) \ \text{a.e.}$$

is convex for $\nu \in [0,\infty)$.

Proof. For $\nu = 0$ the function $|\cdot|_{[0]} = |\cdot|$ is convex by the triangle inequality while for $\nu > 0$ $|\cdot|_{[\nu]}$ is convex since its Hessian is positive definite. Now consider feasible $\mathcal{T}_1$ and $\mathcal{T}_2$ and fix $\nu \in [0,\infty)$ arbitrary. For all $\lambda \in (0,1), \bar{\lambda} = 1 - \lambda$ we have

$$\begin{aligned} |\nabla(\lambda \mathcal{T}_1(x,z) + \bar{\lambda}\mathcal{T}_2(x,z))|_{[\nu]} &\leq \lambda|\nabla \mathcal{T}_1(x,z)|_{[\nu]} + \bar{\lambda}|\nabla \mathcal{T}_2(x,z)|_{[\nu]} \\ &\leq \lambda(\mathcal{T}_1)_z(x,z)\omega(x,z) + \bar{\lambda}(\mathcal{T}_2)_z(x,z)\omega(x,z) \\ &= (\lambda \mathcal{T}_1(x,z) + \lambda \mathcal{T}_2(x,z))_z\omega(x,z) \end{aligned}$$

□

Next we define the operator

$$\Psi : \mathcal{T} \mapsto \Psi(\mathcal{T}), \quad \Psi(\mathcal{T})(x,z) = |\nabla \mathcal{T}(x,z)|_{[\nu]} \tag{5.9}$$

and find the following result concerning differentiability.

Lemma 5.2.3. *The operator Φ defined in* (5.9) *is Fréchet differentiable from $C^{2,1}(\overline{\mathcal{Q}})$ to $C^{1,\theta}(\overline{\mathcal{Q}})$ for any $\theta \in (0,1/4)$ with derivative*

$$D\Phi(\mathcal{T})[\delta\mathcal{T}] = \frac{\nabla \mathcal{T}}{|\nabla \mathcal{T}|_{[\nu]}} \cdot \nabla \delta \mathcal{T}.$$

Proof. According to Proposition 2.1.1, for any $\theta_1 \in [0,1/2)$ the gradient is a bounded linear operator

$$\nabla : C^{2,1}(\bar{\mathcal{Q}}) \to (C^{1,\theta_1}(\bar{\mathcal{Q}}))^2.$$

Next we show the Fréchet differentiability of

$$|\cdot|_{[\nu]} : (C^{1,\theta_1}(\bar{\mathcal{Q}}))^2 \to C^{1,\theta_2}(\bar{\mathcal{Q}})$$

for certain θ_2 with respect to $\|v\|_{C^{1,\theta_2}(\bar{\mathcal{Q}})} = \|v\|_\infty + \sum_{i=1}^2 \|v_{x_i}\|_\infty + \langle v \rangle_z^{\theta_2}$.
Consider $p, h \in (C^{1,\theta_1}(\bar{\mathcal{Q}}))^2$ and fix some $(x,z) \in \mathcal{Q}$. By Taylor expansion we obtain

$$|p+h|_{[\nu]} - |p|_{[\nu]} - \frac{p}{|p|_{[\nu]}} \cdot h = \int_0^1 \left(\frac{p+lh}{|p+lh|_{[\nu]}} - \frac{p}{|p|_{[\nu]}} \right) \cdot h \ \ dl \tag{5.10}$$

which can be bounded in the $\|\cdot\|_\infty$-norm by

$$\begin{aligned}
\left| \int_0^1 \left(\frac{p+lh}{|p+lh|_{[\nu]}} - \frac{p}{|p|_{[\nu]}} \right) \cdot h \ \ dl \right| &\leq \int_0^1 \left| \frac{p+lh}{|p+lh|_{[\nu]}} - \frac{p}{|p|_{[\nu]}} \right| |h| dl \\
&\leq \int_0^1 cl\, |h|\, |h| dl = c|h|^2 \leq c\|h\|^2_{(C^{1,\theta_1}(\bar{\mathcal{Q}}))^2}.
\end{aligned}$$

Here the second estimate is obtained by the equivalence of norms in finite dimensions and an application of the fundamental theorem of calculus. The derivatives of the remainder term in (5.10) can be estimated as follows. By the regularity of p, h we can differentiate under the integral sign and obtain

$$\begin{aligned}
&\int_0^1 \frac{d}{dx_i} \left[p \cdot h \left(\frac{1}{|p+lh|_{[\nu]}} - \frac{1}{|p|_{[\nu]}} \right) + l \frac{h \cdot h}{|p+lh|_{[\nu]}} \right] dl \\
&= \int_0^1 (\dot{p} \cdot h + p \cdot \dot{h}) \left(\frac{1}{|p+lh|_{[\nu]}} - \frac{1}{|p|_{[\nu]}} \right) + p \cdot h \left(\frac{(\dot{p} + l\dot{h}) \cdot (p+lh)}{|p+lh|_{[\nu]}} - \frac{\dot{p} \cdot p}{|p|_{[\nu]}} \right) \\
&\quad + \left(\frac{2l(\dot{h} \cdot h)|p+lh|^2_{[\nu]} + l(h \cdot h)(\dot{p} + l\dot{h}) \cdot (p+lh)}{|p+lh|^3_{[\nu]}} \right) dl
\end{aligned}$$

where the dot notation represents the derivatives of the corresponding functions with respect to the spatial variables x_i. The absolute value of this expression can be estimated from above by $c\|h\|^2_{(C^{1,\theta_1}(\bar{Q}))^2}$. All of the functions involved are continuous and so the pointwise bounds are valid for all $(x,t) \in \bar{Q}$. Finally we show, that the Hölder seminorm vanishes. p and h are Hölder continuous with respect to z and exponent $\theta_1 \in (0,1/2)$. By Lemma 2.1.1 (4) the remainder defined in (5.10) has the same regularity. Fixing $x \in \Omega$ and considering the remainder for different $z, \tilde{z}$ we obtain

$$\begin{aligned}
&\int_0^1 \left| \left(\frac{p+lh}{|p+lh|_{[\nu]}} - \frac{p}{|p|_{[\nu]}} \right) \cdot h - \left(\frac{\tilde{p}+l\tilde{h}}{|\tilde{p}+l\tilde{h}|_{[\nu]}} - \frac{\tilde{p}}{|\tilde{p}|_{[\nu]}} \right) \cdot \tilde{h} \right|^{\bar{\theta}} \\
&\qquad \cdot \left| \left(\frac{p+lh}{|p+lh|_{[\nu]}} - \frac{p}{|p|_{[\nu]}} \right) \cdot h - \left(\frac{\tilde{p}+l\tilde{h}}{|\tilde{p}+l\tilde{h}|_{[\nu]}} - \frac{\tilde{p}}{|\tilde{p}|_{[\nu]}} \right) \cdot \tilde{h} \right|^{1-\bar{\theta}} dl \\
&\leq \int_0^1 c|z-\tilde{z}|^{\theta_1\bar{\theta}} \left| \left(\frac{p+lh}{|p+lh|_{[\nu]}} - \frac{p}{|p|_{[\nu]}} \right) \cdot h - \left(\frac{\tilde{p}+l\tilde{h}}{|\tilde{p}+l\tilde{h}|_{[\nu]}} - \frac{\tilde{p}}{|\tilde{p}|_{[\nu]}} \right) \cdot \tilde{h} \right|^{1-\bar{\theta}} dl \\
&\leq \int_0^1 c|z-\tilde{z}|^{\theta_1\bar{\theta}} \left(cl|h||h| + cl|\tilde{h}||\tilde{h}| \right)^{1-\bar{\theta}} dt \leq c|t-\tilde{t}|^{\theta_1\bar{\theta}} \|h\|^{2-2\bar{\theta}}_{(C^{1,\theta_1}(\bar{Q}))^2}
\end{aligned}$$

with the same estimates as above for any $\bar{\theta} \in (0,1)$. Thus the Hölder seminorm with respect to z and exponent $\theta_1\bar{\theta}$ vanishes as h does for any $\bar{\theta} < 1/2$. Since $\theta_1 \in (0,1/2)$ this proves the claimed differentiability of $|\cdot|_{[\nu]}$. Now the chainrule for Fréchet derivatives provides the claimed result for the superposition operator. □

5.2.2 Characteristics of the Partial Differential Equation

In this section we will discuss parameters of the underlying partial differential equation to formulate a well posed initial-value-boundary-value problem. For the application of existence and stability results, the first order differential operator

$$H(x,z,q) = -\frac{1}{\omega(x,z)}|q|_{[\nu]} - u(x,z)$$

has to be Lipschitz continuous with respect to all of its arguments. This is satisfied if $1/\omega(x,z)$ and u have this regularity. According to (5.2) ω is strictly bounded away from zero and thus Lipschitz continuity of ω for the desired property (see Lemma 2.1.1). In the following, we assume $\omega \in C^{1+\theta}(\overline{Q})$ with $\theta \in (0,0.5)$ satisfying the assumption of strict positivity $\omega \geq \omega_0 > 0$ and a pointwise equality condition $\omega(x,0) = \omega_0$ on $S := \partial\Omega \times \{0\}$.
If ω does not meet this requirements, we can replace it by an approximation obtained as solution of the following optimization problem.

$$\begin{aligned}
\min_{\tilde{\omega} \in H^3(Q)} \frac{1}{2}\|\omega - \tilde{\omega}\|^2_{L^2(Q)} &+ \frac{\alpha}{2}\|\tilde{\omega}\|^2_{H^3(Q)} = \mathcal{J}(\tilde{\omega}) \\
\text{s.t. } \omega_0 &\leq \tilde{\omega}(x,z) \;\; \forall (x,z) \in Q \qquad (5.11) \\
\omega_0 &= \tilde{\omega} \quad \text{on } S
\end{aligned}$$

Lemma 5.2.4. *Problem* (5.11) *has a unique solution with bounded first order derivatives.*

Proof. For (5.11), the feasible set

$$\mathcal{D} = \{v \in H^3(\mathcal{Q}) | v \geq \omega_0 \;\; a.e., v = \omega_0 \text{ on } S\}$$

is closed and nonempty since $\hat{\omega} \equiv \omega_0$ is feasible. In addition it is convex and thus weakly closed (Theorem A.4.1). By the norm character, the objective is bounded from below and coercive.
Let $\omega_n \in \mathcal{D}$ denote an infimizing sequence for the problem such that

$$\lim_{n\to\infty} \mathcal{J}(\omega_n) = \inf_{\omega\in\mathcal{D}} \mathcal{J}(\omega) = a.$$

By Theorem A.2.1, any element of the sequence is an element of $C^{1+\theta}(\overline{\mathcal{Q}})$ and the condition to the value on the boundary is well defined. The coercivity of the objective implies, that the infimizing sequence is bounded in $H^3(\mathcal{Q})$ and thus contains a weakly converging subsequence indexed by n with

$$\omega_n \rightharpoonup \tilde{\omega} \text{ in } H^3(\mathcal{Q}).$$

Theorem A.2.1 yields strong convergence in $C^{1+\theta}(\overline{\mathcal{Q}})$ along a further subsequence ω_n. The involved norms are continuous and weakly lower continuous respectively and we obtain

$$a = \liminf_{n\to\infty} \mathcal{J}(\omega_n) \geq \mathcal{J}(\tilde{\omega}) \geq a.$$

Consequently, the infimum is attained. Uniqueness follows from the strict convexity of the objective. Next we consider

$$\alpha\|\tilde{\omega}\|_{C^1(\overline{\mathcal{Q}})} \leq \alpha\|\tilde{\omega}\|_{C^{1+\theta}(\overline{\mathcal{Q}})} \leq \alpha\|\tilde{\omega}\|_{H^3(\mathcal{Q})} \leq \|\omega - \hat{\omega}\|_{L^2(\mathcal{Q})} + \alpha\|\hat{\omega}\|_{H^3(\mathcal{Q})}$$

providing $\|\tilde{\omega}\|_{C^1(\mathcal{Q})} \leq \alpha^{-1}[\|\omega\|_{L^2(\mathcal{Q})} + (1+\alpha)\underline{\omega}|\Omega|]$. □

Now we fix $\theta \in (0, 0.5)$ for the remainder of this chapter.
For the application of the theory for viscosity solutions from Chapter 4 we need a domain whose boundary is of certain regularity, i.e. it is locally a $C^{2+\theta}$-function. This requirement may already be met by the original domain of the open pit mine. However, the proper formulation of boundary conditions for the partial differential equation (5.8) and its approximations discussed below, need a slightly smaller domain than the original one. Consequently, we consider an open set $\Omega^\mu \subset \Omega$ satisfying the following properties. First, the boundary of Ω^μ is locally the graph of a $C^{2+\theta}$-function. Second, for any $x \in \partial\Omega^\mu$ we have

$$\mu \geq dist(x, \partial\Omega) \geq \underline{\mu} > 0$$

for some $\mu > 0$. The only purpose of $\underline{\mu}$ is to ensure that $\partial\Omega^\mu$ is bounded away from $\partial\Omega$ and thus one might choose $\underline{\mu} = \mu/2$. Note that if the boundary of Ω is sufficiently smooth, one might define Ω^μ as $x \in \Omega$ such that $dist(x, \partial\Omega) > \mu$ for μ small enough. This is based on the result, that for $k \geq 2$ the distance function to the boundary of a set that is locally a C^k-function is a k times differentiable function in a neighborhood of the boundary (see, e.g. [57, Appendix]).
On the lateral boundary of the space - depth - cylinder, the evolution of the time labeling

function will now be defined by a function reaching a particular value T at a depth of μ. We assume, that the material in $(\Omega \backslash \Omega^\mu) \times [0, \overline{z}]$ is hard enough to allow the interpolation of the TLF's from the boundary $\partial\Omega^\mu$ to the actual boundary $\partial\Omega$.
Since the TLF assigns the time of excavation to each point in the deposit, the initial value on $\overline{\Omega}$ has to be zero. The condition on $\partial\Omega \times Z$ is less obvious and strongly depends on the introduction of the perturbed domain Facing the second point in definition 5.1 the function which determines the behavior of the TLF on the boundary $\partial\Omega \times [0, \bar{z}]$ would be forced to have the form

$$\Psi(x, z) = \begin{cases} 0 & z = 0 \\ \eta(z) & z > 0 \end{cases}$$

with $\eta(z) \geq T$ for all $z \in (0, \bar{z}]$ since material in the lateral boundary will never be excavated. The suggested function is not even continuous which is crucial for the regularity of solutions to the Hamilton Jacobi Equation. For the perturbed domain Ω^μ we can define a function Ψ that is differentiable on an open neighborhood of $\partial\Omega^\mu \times [0, \overline{z}]$ and monotonically increasing with respect to the vertical coordinate. Moreover it is chosen to satisfy $\Psi_z(z) \geq \nu$ reflecting, that the TLF can not be faster on the boundary than induced by the model. One possible choice would be

$$\Psi(x, z) = \left(\frac{T}{\mu^2} - \frac{\nu}{\underline{\omega}\mu}\right) z^2 + \frac{\nu}{\underline{\omega}} z$$

for $z \in [0, \mu]$ smoothly changing into a function of linear growth for $z > \mu$. This function forces the TLF to attain the value T at a depth of μ.
An important question for the mining operation is the overall operating time of the mine. The next results present a possibility to derive an approximation of this quantity in the setting of continuous functions.
Even for the block model and the related Integer- or Mixed Integer Programming approaches for the Open Pit Mine Planning problem it is known, that the so called **ultimate pit** represents a shape of the mine which does not allow any further excavation activities without violating the slope- or stability constraint. Here we present a possibility to either compute this profile directly or at least a lower bound for it. Consider the auxiliary function

$$\overline{\omega}(x) = \max\{\omega(x, z) | z \in [0, \overline{z}]\}.$$

Since ω is continuously differentiable, $\overline{\omega}$ has to be continuous on $\overline{\Omega}$. From (5.3) we obtain, that any stable profile p satisfies

$$|\nabla p(x)| \leq \omega(x, p(x)) \leq \overline{\omega}(x).$$

The following result is a consequence of [108, Theorem 5.3].

Proposition 5.2.1. *There exist a unique Lipschitz continuous viscosity solution p_u for*

$$|\nabla p_u(x)| = \overline{\omega}(x) \ \text{on} \ \Omega^\mu, \quad p_u(x) = \mu \ \text{on} \ \partial\Omega^\mu. \tag{5.12}$$

Moreover, the solution is the maximum element of the set

$$\tilde{\mathcal{P}} = \{v \in W^{1,\infty}(\Omega^\mu) | |\nabla v| \leq \overline{\omega} \ a.e. \ in \ \Omega^\mu, v \leq \mu \ on \ \partial\Omega^\mu\}$$

By the latter property we obtain $|\nabla p| \leq |\nabla p_u|$ for any stable profile p and thus found an upper bound for all feasible profiles by the last property of the preceding result.

The Lipschitz continuity of p_u implies, that (5.12) holds almost everywhere. So the lower bound is a generalized solution of the Eikonal equation. If $\omega(x,z)$ is concave and nonincreasing with respect to z, we can apply [108, Theorem 5.4] to ensure the existence of a unique viscosity solution of

$$|\nabla p_u(x)| = \omega(x, p_u(x)) \text{ on } \Omega^\mu, \quad p_u(x) = \mu \text{ on } \partial\Omega^\mu$$

providing a sharper lower bound for the ultimate pit. However, both requirements on ω are highly unlikely to be met as, in terms of mining, this implies that material softens with increasing depth.

The property of being the maximal element of $\tilde{\mathcal{P}}$ is similar to the result [4, Proposition 3]. By the ultimate pit we can answer the question for an upper bound of the total time any TLF can spend on physically stable profiles. As already carried out, this value is also important for the conditions of the lateral boundary of $\mathcal{Q}$.

Lemma 5.2.5. *An upper bound for the maximal time T of any physically stable excavation process is given as the unique solution of*

$$\int_0^T \kappa(s)ds = \int_\Omega \int_0^{p_u(x)} e(x,s)ds.$$

This holds due to the fact, that the ultimate pit represents a pointwise upper bound on every stable profile.

Concerning the boundary conditions we have to take into account the so called compatibility condition. When considering regularized parabolic problems in the upcoming section, classical solutions to the resulting PDE's only exist if the following equation is satisfied.

$$\Psi_z - \varepsilon\Delta u_0 - \frac{1}{\omega}|\nabla u_0|_{[\nu]} = u \text{ on } S. \tag{5.13}$$

Here the condition $\omega \equiv \omega_0$ on S from Lemma 5.2.4 enters. By $u_0 \equiv 0$ which is the natural initial condition for the time labeling functions and the properties of Ψ this introduces the additional constraint $u \equiv 0$ in S to the set of admissible slack functions u.

The last quantity we have to define is $\bar{z}$. Since we have to ensure, that all admissible time labeling functions exceed T everywhere in Ω^μ, we have to consider the maximal speed, a TLF can attain in the vertical direction and choose $\bar{z}$ such that this attains T still in the volume. Fortunately, this value is defined by ν and we obtain

$$\bar{z} = T/\nu.$$

So far the model is only defined for an initial profile of the mine which is equal to zero. Minor adjustment of the involved quantities allows for the consideration of arbitrary, sufficiently regular initial shapes of the open pit mine. In this case we consider the transformation $\tilde{p} = p_0 + p$ and $\mathcal{T}(x, \tilde{p}(x))$ represents the time which is needed to get from p_0 to the actual profile. The state equation (5.8) now becomes

$$\mathcal{T}_z - \frac{1}{|\nabla p_0(x)|_{[\tilde{\nu}]} + \omega}|\nabla \mathcal{T}| = u$$

where we had to consider the perturbed norm again to obtain regularity of $1/(|\nabla p_0(x)|_{[\tilde{\nu}]} + \omega)$. Note that $\tilde{\nu}$ can be chosen independently from ν introduced above. Thus we only have to correct u on S such that (5.13) is still satisfied and can concentrate on the case $p_0 \equiv 0$ without loss of generality.
For notational simplicity we redefine $\Omega = \Omega^\mu$ for the remainder of this chapter unless indicated otherwise.

5.2.3 The Approximating Parabolic Partial Differential Equations

(PDE^0) is a first order hyperbolic partial differential equation as discussed in Chapter 4. Note that in this case the parameter controlling the evolution does not represent time but the vertical coordinate z.
Thus we have to consider a certain class of solutions to this problem. Since we are looking for Lipschitz continuous functions, we concentrate on viscosity solutions. This concept satisfies the comparison principle, thus also fitting to the model because larger initial values imply larger values of the TLF.
Since viscosity solutions to Hamilton Jacobi equations can be considered as the limit of solutions to corresponding parabolic equations under certain conditions, we will also study those problems. The semilinear parabolic partial differential equation approximating (PDE^0) are given as

$$\begin{aligned} \mathcal{T}_z(x,z) - \varepsilon\Delta\mathcal{T}(x,z) - \tfrac{1}{\omega(x,z)}|\nabla\mathcal{T}(x,z)|_{[\nu]} &= u(x,z) && \text{in } \mathcal{Q} \\ \mathcal{T}(x,z) &= \Psi(z) && \text{on } \partial\Omega \times [0,\overline{z}] \\ \mathcal{T}(x,0) &= 0 && \text{on } \Omega \end{aligned} \tag{PDE$^\varepsilon$}$$

We restrict the considered right hand sides to the set of *admissible controls*

$$U_{ad} = \{u \in C^{1+\theta}(\overline{\mathcal{Q}}) | u(x,z) \geq 0 \text{ in } \mathcal{Q}, u = 0 \text{ on } S\}. \tag{5.14}$$

Let $\varepsilon > 0$ be fixed. For (PDE$^\varepsilon$) we find the following result.

Proposition 5.2.2. *For every $\varepsilon, \nu > 0$ and $u \in U_{ad}$ there exist a unique solution $\mathcal{T}$ of* (PDE$^\varepsilon$) *with $\mathcal{T} \in C^{2,1}(\overline{\mathcal{Q}})$.*

Proof. Recall the regularity of Ω and (5.13). The proof is direct application of Theorem 2.2.2. and we have to verify assumptions 1) to 5). In the setting of this theorem we have

$$a(x,z,\mathcal{T},q) = \frac{1}{\omega(x,z)}|q|_{[\nu]}$$

satisfying 1) for $\beta_1 = 1$ and $\beta_2 = \nu^2/(4\underline{\omega})$. For 2) we utilize the equivalence of norms in $\mathbb{R}^n$, $\nu < 1$ and basic estimates to obtain

$$\varepsilon\left(\sum_{i=1}^n |q_i|\right)(1 + |q|_{\mathbb{R}^n}) + \frac{1}{\omega}|q|_{[\nu]} \leq (\varepsilon\tilde{c} + 1/\underline{\omega})(1 + |q|_{\mathbb{R}^n})^2$$

where $\tilde{c}$, the constant in $|q|_{l^1} \le \tilde{c}|q|_{l^2}$, depends on n. By assumption, ω is Lipschitz continuous with constant L_ω and strictly positive. Thus $1/\omega$ is Lipschitz with constant $L_\omega/\underline{\omega}^2$. Consequently estimate 3) is satisfied for $\max\{c_1/(\underline{\omega}\nu), L_\omega/\underline{\omega}^2\}$. Finally, the regularity requirements are met by construction of the functions involved. □

Note that the existence Theorem provides an even higher regularity of the solutions, namely $\mathcal{T} \in C^{2+\theta,1+\theta/2}(\overline{\mathcal{Q}})$. Let $\mathcal{S} : U_{ad} \to C^{2,1}(\overline{\mathcal{Q}})$ denote the control-to-state or solution operator

$$\mathcal{S}(u) := \mathcal{T}$$

where $\mathcal{T}$ is the unique solution of (PDE$^\varepsilon$) with respect to the data u.

Lemma 5.2.6. *Let $u^1, u^2 \in U_{ad}$ be admissible data and set $\mathcal{T}^1 = \mathcal{S}(u^1), \mathcal{T}^2 = \mathcal{S}(u^2)$. Then*

$$\|\mathcal{T}^1 - \mathcal{T}^2\|_{C^{2,1}(\overline{\mathcal{Q}})} \le c\|u^1 - u^2\|_{C^{1+\theta}(\overline{\mathcal{Q}})}$$

with c depending on $\max\{\|u^1\|_{C^{1+\theta}(\overline{\mathcal{Q}})}, \|u^2\|_{C^{1+\theta}(\overline{\mathcal{Q}})}\}$.

Proof. $\mathcal{T}^1$ and $\mathcal{T}^2$ solve (PDE$^\varepsilon$). According to Remark 2.2.1 the spatial gradients of solutions to (PDE$^\varepsilon$) are bounded for all $\hat{u}$ with $\|u^1 - \hat{u}\|_{C^{1+\theta}(\overline{\mathcal{Q}})} \le \|u^1 - u^2\|_{C^{1+\theta}(\overline{\mathcal{Q}})}$. Subtracting (PDE$^\varepsilon$) for data u^1, u^2 and abbreviating $v = \mathcal{T}^1 - \mathcal{T}^2, \delta u = u^1 - u^2$ provides

$$\begin{array}{rll} v_z(x,z) - \varepsilon\Delta v(x,z) - \frac{1}{\omega(x,z)}(|\nabla\mathcal{T}^1(x,z)|_{[\nu]} - |\nabla\mathcal{T}^2(x,z)|_{[\nu]}) & = \delta u(x,z) & \text{in } \mathcal{Q} \\ v(x,z) & = 0 & \text{on } \Gamma \end{array} . \tag{5.15}$$

For any $(x,z) \in \overline{\mathcal{Q}}$ the function $f : \mathbb{R}^2 \to \mathbb{R}$, $f(q) = |q|_{[\nu]}$ is continuously differentiable with respect to its arguments. The fundamental Theorem of calculus implies

$$|\nabla\mathcal{T}^1(x,z)|_{[\nu]} - |\nabla\mathcal{T}^2(x,z)|_{[\nu]} = \int_0^1 \frac{d}{dl}|l\nabla\mathcal{T}^1(x,z) + (1-l)\nabla\mathcal{T}^2(x,z)|_{[\nu]}dl$$

proving the equivalence of (5.15) to the linear Cauchy Problem

$$\begin{array}{rll} v_z - \varepsilon\Delta v - \left[\frac{1}{\omega}\int_0^1 \frac{l\nabla\mathcal{T}^1 + (1-l)\nabla\mathcal{T}^2}{|l\nabla\mathcal{T}^1 + (1-l)\nabla\mathcal{T}^2|_{[\nu]}}dl \cdot \nabla v\right] & = \delta u & \text{in } \mathcal{Q} \\ v & = 0 & \text{on } \Gamma \end{array} .$$

According to Lemma 2.1.1 the integral term is Hölder continuous with constant $\tilde{\theta}$. Usual estimates for this class of problems as Theorem 2.2.1 yield

$$\|v\|_{C^{2+\tilde{\theta},1+\tilde{\theta}/2}(\overline{\mathcal{Q}})} \le c\|\delta u\|_{C^{\tilde{\theta},\tilde{\theta}/2}(\overline{\mathcal{Q}})} \le c\|\delta u\|_{C^{\theta}(\overline{\mathcal{Q}})} \tag{5.16}$$

for some $0 < \tilde{\theta} < \theta$ and c depending on the pointwise difference of the gradients. Now the claim has been proven as there exists a solution only depending on the norm of the right hand side for bounded δu since (5.16) implies $\|v\|_{C^{2,1}(\overline{\mathcal{Q}})} \le \tilde{c}\|\delta u\|_{C^{1+\theta}(\overline{\mathcal{Q}})}$. □

After establishing local Lipschitz continuity of $\mathcal{S}$ with respect to the data we will investigate its smoothness properties. The proof follows the line of [149] and is adapted to the setting we are considering.

Proposition 5.2.3. *The control-to-state operator $\mathcal{S} : U_{ad} \to C^{2,1}(\overline{\mathcal{Q}})$ is Fréchet differentiable. At a given $\bar{u}$ the derivative in direction h, $D\mathcal{S}(\bar{u})[h]$, is given as solution q of*

$$\begin{aligned} q_z - \varepsilon\Delta q - \tfrac{\nabla\bar{u}}{\omega|\nabla\bar{u}|_{[\nu]}}\cdot\nabla q &= h \quad \text{in } \mathcal{Q} \\ q &= 0 \quad \text{on } \Gamma \end{aligned} \tag{5.17}$$

Proof. According to Theorem 2.2.1 (5.17) attains a unique solution $q \in C^{2,1}(\overline{\mathcal{Q}})$ (actually $C^{2+\theta,1+\theta/2}(\mathcal{Q})$ for $h \in C^{\theta}(\mathcal{Q})$). Consider (PDE$^{\varepsilon}$) for $\bar{u}+h, \bar{u}$ with $\mathcal{T}^1 = \mathcal{S}(\bar{u}+h), \mathcal{T}^2 = \mathcal{S}(\bar{u})$. The difference of the state equations is given as

$$\begin{aligned} v_z - \varepsilon\Delta v - \tfrac{1}{\omega}(|\nabla\mathcal{T}^1(x,z)|_{[\nu]} - |\nabla\mathcal{T}^2(x,z)|_{[\nu]}) &= h \quad \text{in } \mathcal{Q} \\ v &= 0 \quad \text{on } \Gamma \end{aligned}$$

with $v = \mathcal{T}^1 - \mathcal{T}^2$. By Lemma 5.2.3, the superposition operator is Fréchet differentiable from $C^{2,1}(\bar{\mathcal{Q}})$ to $C^{1,\theta}$ and we can use the expansion

$$|\nabla\mathcal{T}^1|_{[\nu]} - |\nabla\mathcal{T}^2|_{[\nu]} = \frac{\nabla\mathcal{T}^1}{|\nabla\mathcal{T}^1|_{[\nu]}}\cdot\nabla v + r$$

for a function r defined by the integral in (5.10) and satisfying $\|r\|_{C^{1,\theta}(\overline{\mathcal{Q}})}/\|v\|_{C^{2,1}(\overline{\mathcal{Q}})} \to 0$ for $h \to 0$. The identity $v = q + \mathcal{T}^{\rho}$ with q satisfying (5.17) provides a linear parabolic PDE defining $\mathcal{T}^{\rho}$ according to

$$\begin{aligned} -\varepsilon\Delta\mathcal{T}^{\rho} - \tfrac{1}{\omega}\tfrac{\nabla\mathcal{T}^1}{|\nabla\mathcal{T}^1|_{[\nu]}}\cdot\nabla\mathcal{T}^{\rho} &= -r \quad \text{in } \mathcal{Q} \\ \mathcal{T}^{\rho} &= 0 \quad \text{on } \Gamma \end{aligned}\,.$$

Recall that Proposition 5.2.2 provides a higher regularity of solutions, i.e.

$$\mathcal{T}^1, \mathcal{T}^2 \in C^{2+\theta,1+\theta/2}(\overline{\mathcal{Q}})$$

for θ denoting the Hölder exponent of $\bar{u}$ and $\bar{u}+h$. Using this regularity and the calculus for Hölder exponents (see Lemma 2.1.1), we find r to be Hölder continuous with exponent $0 < \tilde{\theta} \le \theta$ guaranteing $\mathcal{T}^{\rho} \in C^{2,1}(\overline{\mathcal{Q}})$ by Theorem 2.2.1. The embedding theory for Hölder spaces ensures that $C^{\theta}(\overline{\mathcal{Q}}) \to C^{\tilde{\theta}}(\overline{\mathcal{Q}})$ is continuous for all $0 \le \tilde{\theta} \le \theta \in (0,1)$. From [98, Theorem 2.1] we further obtain

$$\|\mathcal{T}^{\rho}\|_{C^{2,1}(\overline{\mathcal{Q}})} \le c\|r\|_{C^{\tilde{\theta}}(\overline{\mathcal{Q}})}.$$

Finally estimate (5.16) provides

$$\frac{\|\mathcal{T}^{\rho}\|_{C^{2,1}}}{\|h\|_{C^{1+\theta}}} = \underbrace{\frac{\|\mathcal{T}^{\rho}\|_{C^{2,1}}}{\|r\|_{C^{1,\theta/2}}}}_{\le c}\underbrace{\frac{\|r\|_{C^{1,\theta/2}}}{\|v\|_{C^{2,1}}}}_{\to 0}\underbrace{\frac{\|v\|_{C^{2,1}}}{\|h\|_{C^{1+\theta}}}}_{\le c}$$

for $\|h\|_{C^{1+\theta}(\overline{\mathcal{Q}})} \to 0$ implying $\|\mathcal{T}^{\rho}\|_{C^{2,1}(\overline{\mathcal{Q}})} = o(\|h\|_{C^{1+\theta}(\overline{\mathcal{Q}})})$ and providing, that

$$\mathcal{S}(\bar{u}+h) - \mathcal{S}(\bar{u}) = \mathcal{T}^1 - \mathcal{T}^2 = D\mathcal{S}(\bar{u})[h] + \mathcal{T}^{\rho} = D\mathcal{S}(\bar{u})[h] + r(\bar{u},h)$$

hols with $r(\bar{u}, h) = \mathcal{T}^\rho$ enjoying the properties needed for Fréchet differentiability. $\square$

Finally we discuss convergence properties of solutions $\mathcal{T}_\varepsilon$ for the underlying (PDE$^\varepsilon$) when the viscosity parameter tends to zero. Recalling Theorem 4.1.1, we have to ensure, that $\mathcal{T}_\varepsilon$ converge strongly with respect to the supremum norm to some limit $\mathcal{T}$ in $C(\mathcal{Q})$. This will be achieved by Theorem 2.1.1. Thus we have to prove, that all involved functions are Lipschitz continuous with a common Lipschitz constant (see Lemma A.1.1). It is well known (see e.g. [40, 41, 108]), that if the spatial gradients $\nabla\mathcal{T}_\varepsilon$ are bounded independently of ε, this requirement is met. Under suitable assumptions we obtain the following result.

Proposition 5.2.4. *Let u be fixed, consider a sequence $\varepsilon \to 0$ and a family of solutions $\mathcal{T}^\varepsilon$ to* (PDE$^\varepsilon$) *satisfying*

$$|\mathcal{T}^\varepsilon_z| \le c_z \text{ and } |\mathcal{T}^\varepsilon_{x_i}| \le c_x \text{ for } i = 1, 2.$$

Then there exist a function $\mathcal{T} \in C(\overline{\mathcal{Q}})$ and a subsequence $\varepsilon \to 0$ such that $\mathcal{T}^\varepsilon \to \mathcal{T}$ uniformly in $\mathcal{Q}$. Moreover, $\mathcal{T}$ is Lipschitz continous and satisfies

$$\mathcal{T}_z - \frac{1}{\omega(x,z)}|\nabla\mathcal{T}|_{[\nu]} = u \tag{5.18}$$

pointwise almost everywhere.

Proof. The bounds on $\mathcal{T}^\varepsilon$ imply uniform convergence in $C(\overline{\mathcal{Q}})$ of the corresponding $\mathcal{T}^\varepsilon$ according to Arzela Ascoli. Theorem 4.1.1 yields, that the limit element is a viscosity solution of (5.18).
The first order derivatives of $\mathcal{T}^\varepsilon$ are bounded in $L^\infty(\mathcal{Q})$ due to the continuous embedding $C(\overline{\mathcal{Q}}) \to L^\infty(\mathcal{Q})$. Consequently, they contain a weak-* converging subsequence with corresponding limit elements $\hat{\mathcal{T}}_z, \hat{\mathcal{T}}_{x_1}$ and $\hat{\mathcal{T}}_{x_2}$, bounded by the same constants as the partial derivatives of the $\mathcal{T}^\varepsilon$ (see, e.g. [157]). The subsequence combining all convergence properties provides for an arbitrary $\varphi \in C_c^\infty(\mathcal{Q})$, all $\varepsilon > 0$ and $s \in \{x_1, x_s, z\}$

$$\int\limits_{\mathcal{Q}} \mathcal{T}^\varepsilon_s \varphi = -\int\limits_{\mathcal{Q}} \mathcal{T}^\varepsilon \varphi_s$$

For $\varepsilon \to 0$ this yields

$$\int\limits_{\mathcal{Q}} \hat{\mathcal{T}}_s \varphi = -\int\limits_{\mathcal{Q}} \mathcal{T} \varphi_s$$

Consequently $\hat{\mathcal{T}}_z, \hat{\mathcal{T}}_{x_1}, \hat{\mathcal{T}}_{x_2}$ are the weak derivatives of $\mathcal{T}$. Thus $\mathcal{T}$ is Lipschitz (see Theorem A.1.1). By Remark 4.1.1, (5.18) is satisfied almost everywhere. $\square$

5.2.4 The Effort Constraint

In this section we discuss the effort constraint (5.4). Therefore we define

$$E_{\mathcal{T}}(t) := \int\limits_{\Omega} \int_0^{p_t} e(x,s)\,ds\,dx \tag{5.19}$$

where p_t is the implicit profile function defined in Lemma 5.2.1.

Proposition 5.2.5. *For $\mathcal{T} \in W^{1,\infty}(\mathcal{Q})$ with $\mathcal{T}_z \geq \nu$ a.e., (5.4) is satisfied if*

$$\int\limits_{\Omega} \frac{e(x, p_t(x))}{\mathcal{T}_z(x, p_t(x))} dx \leq c(t)$$

holds for almost all times $t \in (0, T)$.

Proof. Given Lipschitz continuous $\mathcal{T}$ with $\mathcal{T}_z \geq \nu$, the effort functional $E_{\mathcal{T}}(t)$ is Lipschitz continuous in t since basic manipulations provide

$$\begin{aligned} |E_{\mathcal{T}}(t_2) - E_{\mathcal{T}}(t_1)| = \left| \int_{\Omega} \int_{p_{t_1}(x)}^{p_{t_2}(x)} e(x,s) ds dx \right| &\leq \|e\|_{L^\infty} \int_{\Omega} |p_{t_2}(x) - p_{t_1}(x)| \\ &\leq \|e\|_{L^\infty} \frac{|\Omega|}{\nu} |t_2 - t_1| \end{aligned} \tag{5.20}$$

In addition, $\hat{e}(x, p_t(x))$, defined by the identity

$$E_{\mathcal{T}}(t) = \int_{\Omega} \hat{e}(x, p_t(x)) dx = \int_{\Omega} \int_0^{p_t(x)} e(x,s) ds dx$$

is integrable over Ω for all $t \in [0, T]$ and is, for almost all $x \in \Omega$, an absolutely continuous function in t. The pointwise derivative

$$\frac{d}{ds} \hat{e}(x, s) = \frac{e(x, p_s(x))}{\mathcal{T}_z(x, p_s(x))}$$

exists almost everywhere and is locally bounded. Consequently we can differentiate under the integral sign and obtain

$$\frac{d}{ds} E_{\mathcal{T}}(s) = \int\limits_{\Omega} \frac{e(x, p_s(x))}{\mathcal{T}_z(x, p_s(x))} dx.$$

As $C(\tau) = \int_0^\tau c(s) ds$ is absolutely continuous, the derivative at τ equals $c(\tau)$ almost everywhere. The claimed relation surely is sufficient for the capacity constraint to hold. □

Differentiability properties for (5.4) can only be proven for solutions of the approximating parabolic equations since the increased regularity of the states is crucial in the proofs. Besides the already introduced properties of e we in addition assume that it behaves locally nice in the following sense.

Assumption 5.1. *Let e be piecewise differentiable with uniformly bounded derivatives. Moreover, let the set*

$$\mathcal{Q}_- = \{(x, z) \in \overline{\mathcal{Q}} | e \text{ is not differentiable in } (x, z)\}$$

be a finite collection of two dimensional manifolds $\{M_i\}_{i=1}^m$ with boundary.

In particular, this assumption is satisfied, if the data are provided by a block model of the deposit and piecewise constant on each of the blocks. In the following auxiliary Lemma, the term n-measure denotes the measure on an n dimensional manifold. The proof uses arguments suggested by Professor Schüth.

Lemma 5.2.7. *Let a TLF $\mathcal{T}$ be given such that the inverse function $p_t : \Omega \to [0, \bar{z}]$ defined by $\mathcal{T}(x, p_t(x)) = t$ is unique and locally Lipschitz continuous. Moreover, consider a set $\Theta \subset \mathcal{Q}$ of Lebesgue 3-measure zero. Then the Lebesgue 1-measure of*

$$\Theta_L = \{t \in (0,T) | p_t(\Omega) \cap \Theta \text{ has positive Lebesgue 2-measure } \}$$

is zero.

Proof. Consider the mapping $Y : \mathcal{Q} \to \Omega \times \mathbb{R}$ defined by

$$Y(x,z) = (x, \mathcal{T}(x,z))$$

By construction Y is Lipschitz and maps sets of Lebesgue measure zero on sets of Lebesgue measure zero. Thus $Y(\Theta)$ has Lebesgue 3-measure zero as well. Enlarging $Y(\Theta)$ to a set of Borel-Measure zero (intersecting it with a countable collection of containing open sets) $\tilde{Y}(\Theta)$. Fubini (see, e.g. [49]) implies, that the set

$$\Theta = \{t \in (0,T) | Y(x, p_t(x)) \cap \tilde{Y}(\Theta) \text{ has positive Borel 2-measure } \}$$

has Borel 1-measure zero and consequently Lebesgue 1-measure zero. Now take any t_0 from the complement Θ^C. Moreover, choose a dense countable subset $\mathbb{X} \in \Omega$. By the local Lipschitz continuity of the implicit function $p_{t_0}(x)$ we find for every $x \in \Omega$ an open neighborhood $\mathcal{O}_x$ such that p_{t_0} is Lipschitz continuous in this neighborhood. Now, for every $x \in \mathbb{X}$ consider the mapping $\hat{Y} : \mathcal{O}_x \times \mathbb{R} \to \mathcal{Q}$

$$\hat{Y}_x(x, t_0) = (x, p_{t_0}(x))$$

which is Lipschitz continuous since p_{t_0} is. It maps $Y(\Theta) \cap (\mathcal{O}_x \times \{t_0\})$, which is a set of Lebesgue 2-measure zero, to $\Theta \cap \{(\tilde{x}, \mathcal{T}^{-1}(\tilde{x})(t_0))$ s.t. $\tilde{x} \in \mathcal{O}_x\}$ which is then of Lebesgue 2-measure zero as well. Since $\mathbb{X}$ was chosen to be dense and countable we obtain

$$\Theta \cap (x, p_{t_0}(x)) = \bigcup_{x \in \mathbb{X}} \left[\Theta \cap \{(\tilde{x}, \mathcal{T}^{-1}(\tilde{x})(t_0)) \text{ s.t. } \tilde{x} \in \mathcal{O}_x\} \right]$$

which by fundamental properties of the Lebesgue measure implies, that the Lebesgue 2-measure of $\Theta \cap (x, p_{t_0}(x))$ is zero. □

The proof given above uses the Lipschitz continuity of $\mathcal{T}$ to construct Lipschitz continuous mappings transforming the excavation process of the profile functions as sketched in Figure 5.3 and reversing this process as well.
Before presenting the main result of this section we provide the following auxiliary result.

Lemma 5.2.8. *Let $t \in [0,T]$ be fixed. Set*

$$\mathcal{Q}_-(x) = \mathcal{Q}_- \cap (\{x\} \times [0, \overline{z}])$$

for all $x \in \overline{\Omega}$. Under Assumption 5.1 on e the distance function

$$\rho(x) = \begin{cases} dist(p_t(x), \mathcal{Q}_-(x)) & if \quad \mathcal{Q}_-(x) \neq \emptyset \\ \overline{z} & if \quad \mathcal{Q}_-(x) = \emptyset \end{cases}$$

is lower semicontinuous.

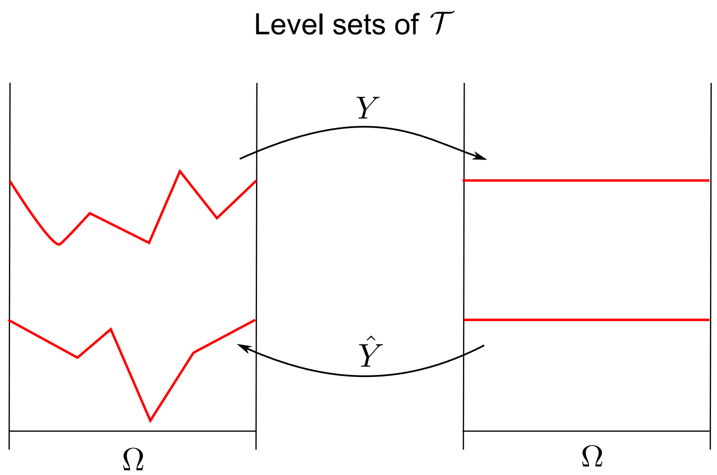


Figure 5.3: Impact of the functions Y and $\tilde{Y}$ in Lemma 5.2.7

Proof. Consider a sequence $\{(x_n, z(x_n))\} \in \mathcal{Q}_-$ with $x_n \to \tilde{x}$ such that $\{(x_n, z(x_n))\} \subset M_i$ for one of the two dimensional manifolds $\mathcal{Q}_-$ consist of. By assumption, M_i is a manifold with boundary and consequently $z(x_n) \to z(\tilde{x})$ has to hold. Thus the limit point is an element of M_i and the mapping

$$\rho^i(x) = dist(p_t(x), M_i \cap (\{x\} \times [0, \overline{z}]))$$

is continuous on $D_i = \{x \in \Omega : \exists z \text{ with } (x, z) \in M_i\}$. Since the distance function can equivalently be defined as

$$\rho(x) = \min_{1 \le i \le m} \rho^i(x)$$

lower semicontinuity follows directly for all $x \in D = \bigcup_{1 \le i \le m} D_i$.
If $x \notin D$ this has to be satisfied for an open neighborhood as well by the closedness of D. Consider a sequence $x_n \to \tilde{x}$ with $\{x_n\} \notin D$ for all n. The limit of the distance function as defined above would be

$$\lim_{n\to\infty} \rho(x_n) = \begin{cases} \overline{z} & \text{if } \tilde{x} \notin D \\ \rho(\tilde{x}) \le \overline{z} & \text{if } \tilde{x} \in D \end{cases}$$

Consequently, the mapping ρ is lower semicontinuous. $\square$

Proposition 5.2.6. *Let $\mathcal{T} \in C^{2,1}(\mathcal{Q})$ with $\mathcal{T}_z \ge \lambda > 0$ and $\mathcal{T}(\cdot, 0) \equiv 0$ be given. Then the Gateaux derivative of $E_\mathcal{T}$ at $\mathcal{T}$ in direction $h \in C^{2,1}(\mathcal{Q})$, satisfying $h(\cdot, 0) \equiv 0$, is given for almost every $t \in (0, T)$ by*

$$D_\mathcal{T} E_\mathcal{T}(t)[h] = -\int_\Omega \frac{e(x, p_t(x))}{\mathcal{T}_z(x, p_t(x))} h(x, p_t(x)) dx.$$

Proof. Consider an arbitrary $t \in (0, T)$ such that

$$\hat{\Omega}_- = p_t(\Omega) \cap \mathcal{Q}_-$$

has Lebesgue 2-measure zero (by Lemma 5.2.7 this holds for almost every $t \in (0, T)$). For arbitrary $x \in \Omega$ and $h \in C^{2,1}(\mathcal{Q})$, we set $z(x) = p_t(x)$ and let $z_h(x)$ denote the unique implicit function solving $\mathcal{T}(x, z_h(x)) + h(x, z_h(x)) = t$ which exist for h sufficiently small. For notational simplicity we omit the argument for both functions if its clear where they are evaluated. They are defined by the identities

$$\begin{array}{rcl} \int_0^z \mathcal{T}_z(x,s)ds &=& t \\ \int_0^{z_h} \mathcal{T}_z(x,s) + h_z(x,s)ds &=& t \end{array} \tag{5.21}$$

Here the second equation admits a unique solution for $\|h\|_{C^{2,1}(\mathcal{Q})}$ small as, by the inverse triangle inequality, we have

$$\mathcal{T}_z + h_z \geq \lambda - \|h\|_{C^{2,1}(\mathcal{Q})} > 0 \tag{5.22}$$

for $\|h\|_{C^{2,1}(\mathcal{Q})} < \lambda$. Then $\mathcal{T} + h$ is strict monotonically increasing in z and (5.21) provides

$$\begin{aligned} \underbrace{\|h\|_{C^{2,1}(\mathcal{Q})}}_{\to 0} \geq |h(x,z)| = \left| \int_0^z h_z(x,s)ds \right| \\ = \left| \int_z^{z_h} \mathcal{T}_z(x,s) + h_z(x,s)ds \right| \geq \underbrace{(\lambda - \|h\|_{C^{2,1}(\mathcal{Q})})}_{>0} |z_h - z| \end{aligned}$$

and consequently, the distance $z_h - z$ vanishes uniformly for $\|h\|_{C^{2,1}(\mathcal{Q})}$ tending to zero. In particular, for any h with $\|h\|_{C^{2,1}(\mathcal{Q})} \leq \lambda/2$, we obtain

$$\|h\|_{C^{2,1}(\overline{\mathcal{Q}})} \geq (\lambda/2)|z_h - z|. \tag{5.23}$$

Now set $\hat{e}(x,j) = \int_0^j e(x,s)ds$ and consider, for $\|h\|_{C^{2,1}(\mathcal{Q})}$ sufficiently small,

$$\begin{aligned} &\left| E_{\mathcal{T}+h}(t) - E_{\mathcal{T}}(t) + \int\limits_{\Omega} \frac{e(x,z)}{\mathcal{T}_z(x,z)} h(x,z)dx \right| = \left| \int\limits_{\Omega} \hat{e}(x,z_h) - \hat{e}(x,z) + \frac{e(x,z)}{\mathcal{T}_z(x,z)} h(x,z)dx \right| \\ &= \left| \int\limits_{\Omega} \int\limits_0^1 \frac{d}{dl}[\hat{e}(x, z + l(z_h - z))] + \frac{e(x,z)}{\mathcal{T}_z(x,z)} h(x,z)dldx \right| \\ &= \left| \int\limits_{\Omega} \int\limits_0^1 e(x, z + l(z_h - z))(z_h - z) + \frac{e(x,z)}{\mathcal{T}_z(x,z)} h(x,z)dldx \right| \end{aligned}$$

where we used that $\hat{e}(x,j)$ is absolutely continuous in j with derivative $e(x,j)$ for almost every

j. Now we add and subtract $e(x,z)(z_h - z)$ and obtain

$$\left|\int_\Omega \int_0^1 [e(x, z + l(z_h - z)) - e(x,z)](z_h - z) + e(x,z)\left[(z_h - z) + \frac{h(x,z)}{\mathcal{T}_z(x,z)}\right] dl dx\right|$$
$$\leq \underbrace{\left(\int_\Omega \int_0^1 |e(x, z + l(z_h - z)) - e(x,z)| dl dx\right)}_{(i)} C\|h\|_{C^{2,1}} + \int_\Omega e(x,z) \underbrace{\left|(z_h - z) + \frac{h(x,z)}{\mathcal{T}_z(x,z)}\right|}_{(ii)} dx$$

In the next step we show, that (ii) is of order $o(\|h\|_{C^{2,1}})$. Here we will use the additional regularity of $\mathcal{T}$ to apply the mean value Theorem. For $\mathcal{T}$ and h given, we obtain for every $x \in \Omega$ that there exists some $\tilde{z}(x) \in (z(x), z_h(x))$ with

$$h(x,z) = \int_0^z h_z(x,s) ds = -\int_z^{z_h} \mathcal{T}_z(x,s) + h_z(x,s) ds = -(\mathcal{T}_z(x, \tilde{z}(x)) + h_z(x, \tilde{z}(x)))(z_h - z).$$

providing

$$z_h - z = -\frac{h(x,z)}{\mathcal{T}_z(x,\tilde{z}(x)) + h_z(x,\tilde{z}(x))}$$

as, by (5.22), $\mathcal{T}_z(x,\tilde{z}(x)) + h_z(x,\tilde{z}(x)) \geq (\lambda/2)\tilde{z}(x)$ for all $\|h\|_{C^{2,1}(\mathcal{Q})} \leq \lambda/2$. With $\tilde{\mathcal{T}}_z = \mathcal{T}_z(x,\tilde{z}(x))$ and $\tilde{h}_z = h_z(x,\tilde{z}(x))$, (ii) can be estimated as

$$\begin{aligned}
&\left|-\frac{h(x,z)}{\tilde{\mathcal{T}}_z + \tilde{h}_z} + \frac{h(x,z)}{\mathcal{T}_z(x,z)}\right| \\
\leq &|h(x,z)| \left|-\frac{1}{\tilde{\mathcal{T}}_z + \tilde{h}_z} + \frac{1}{\mathcal{T}_z(x,z) + \tilde{h}_z} - \frac{1}{\mathcal{T}_z(x,z) + \tilde{h}_z} + \frac{1}{\mathcal{T}_z(x,z)}\right| \\
\leq &|h(x,z)| L |\mathcal{T}_z(x,z) - \tilde{\mathcal{T}}_z| + |h(x,z)| L |\tilde{h}_z| \\
\leq &L2\|h\|_{C^{2,1}} |\mathcal{T}_z(x,z) - \tilde{\mathcal{T}}_z| + L2\|h\|^2_{C^{2,1}}
\end{aligned}$$

The Lipschitz estimates hold since all involved functions are strictly positive due to (5.22) for $\|h\|_{C^{2,1}(\mathcal{Q})} \leq \lambda/2$. Since $\tilde{z}(x) \to z$ uniformly, $|\mathcal{T}_z(x,z) - \tilde{\mathcal{T}}_z| \to 0$ as $\|h\| \to 0$ by continuity of $\mathcal{T}_z$. Thus $(ii) \in o(\|h\|_{C^{2,1}})$.

Now we prove that (i) vanishes as $\|h\|_{C^{2,1}}$ does. Consider the sets

$$\Omega_+^h = \{x \in \Omega | e(x,\cdot) \in C^1((z(x), z_h(x)))\} \text{ and } \Omega_-^h = \Omega \backslash \Omega_+^h.$$

For any $x \in \Omega \backslash \hat{\Omega}_-$ we know $\rho(x) > 0$ and so the triangle inequality yields

$$dist(z(x), \hat{\mathcal{Q}}_-(x)) \leq |z(x) - z_h(x)| + dist(z_h(x), \hat{\mathcal{Q}}_-(x)).$$

Utilizing (5.23) we see, that if $\|h\|_{C^{2,1}} \leq (\rho(x)\lambda)/4$ holds, $|z_h(x) - z(x)| \leq \rho(x)/2$ and $dist(z_h(x), \hat{\mathcal{Q}}_-(x)) \geq \rho(x)/2$. Consequently we find $x \in \Omega_+^h$ and hence, given some $\alpha > 0$, $\|h\|_{C^{2,1}} \leq (\alpha\lambda)/4$ implies, that any $x \in \Theta_\alpha := \{x \in \Omega | \rho(x) > \alpha\}$ is an element of Ω_+^h. By the lower semi continuity of ρ (Lemma 5.2.8), Θ_α is open. Hence its complement Θ_α^C is closed and both sets are measurable. Moreover, Θ_α^C is a super-set for Ω_-^h for $\|h\| = (\alpha\lambda)/4$. Next we observe, that $\Theta^C_{\alpha_1} \subset \Theta^C_{\alpha_2}$ holds for any $\alpha_1 \leq \alpha_2$. Thus the chain of inclusions

$$\Omega^h_- \subset \Theta^C_{4\|h\|/\lambda} \subset \Theta^C_{[1/j]} := \{x \in \Omega | x \in \Theta^C_{4\|h\|/\lambda} \text{ for a } j \in \mathbb{N} \text{ with } 4\|h\|/\lambda \in (1/j, 1/(j+1)]\}$$

is valid. Applying basic estimates we obtain

$$\begin{aligned}
&\int_\Omega \int_0^1 |e(x, z + l(z_h - z)) - e(x,z)| dl dx \\
&= \int_{\Theta_{4\|h\|/\lambda}} \int_0^1 |e(x, z + l(z_h - z)) - e(x,z)| dl dx + \int_{\Theta^C_{4\|h\|/\lambda}} \int_0^1 |e(x, z + l(z_h - z)) - e(x,z)| dl dx \\
&\leq \int_{\Theta_{4\|h\|/\lambda}} L|z_h - z| dx + \int_{\Theta^C_{4\|h\|/\lambda}} 2\|e\|_{L^\infty} dx \leq L|z_h - z||\Omega| + 2\|e\|_{L^\infty} |\Theta^C_{4\|h\|/\lambda}|.
\end{aligned}$$

Here the first term vanishes for $\|h\| \to 0$. Concerning the second term we use the estimate

$$|\Theta^C_{4\|h\|/\lambda}| \leq |\Theta^C_{[1/j]}|$$

holding for certain $j \in \mathbb{N}$ by monotonicity of the Lebesgue measure. The family of sets $\{\Theta^C_{[1/j]}\}_{j=1}^\infty$ is a countable sequence of nested sets and consequently

$$\lim_{j\to\infty} |\Theta^C_{[1/j]}| = |\bigcap_{j=1}^\infty \Theta^C_{[1/j]}|$$

holds. The countable intersection is characterized by

$$\bigcap_{j=1}^\infty \Theta^C_{[1/j]} = \{x \in \Omega | \rho(x) \leq \frac{4\|h\|}{\lambda j} \text{ for all } j \in \mathbb{N}\} = \hat{\Omega}_-$$

with $|\hat{\Omega}_-| = 0$ by assumption. Consequently the second term vanishes for $\|h\| \to 0$ as well. Since this estimates of

$$\left| E_{\mathcal{T}+h}(t) - E_{\mathcal{T}}(t) + \int_\Omega \frac{e(x,z)}{\mathcal{T}_z(x,z)} h(x,z) dx \right|$$

hold for almost every $t \in (0, T)$, they have to hold for the essential supremum as well. □

Note that there is a weaker version of the mean value theorem available which is applicable for Lipschitz continuous function (see [38]) but does not provide enough information on the derivative of $1/\mathcal{T}$ with respect to z. Next we provide a methodology to transform solutions to the parabolic state equation (PDE$^\varepsilon$) violating (5.4) into feasible time labeling functions. Any monotone transformation of a physically stable $\mathcal{T}$ satisfying a certain growth condition provides physically stable profiles. For an abstract transformation function $\phi : C^{2,1}(\overline{\mathcal{Q}}) \to \mathbb{R}^+$ we define

$$\tilde{\mathcal{T}}(x,z) := \phi(\mathcal{T}(x,z))$$

and obtain the following result.

Lemma 5.2.9. *$\tilde{\mathcal{T}}$ satisfies the stability condition, if ϕ is locally Lipschitz continuous, strict monotonically increasing in its argument and if the derivative satisfies*

$$\phi'(\cdot) \geq 1 \tag{5.24}$$

almost everywhere in $\mathbb{R}^+$.

Proof. By Lipschitz continuity all partial derivatives of $\mathcal{T}$ are defined almost everywhere. From (5.7) we obtain

$$\begin{aligned}\tilde{\mathcal{T}}_z(x,z) - \frac{1}{\omega(x,z)}|\nabla\tilde{\mathcal{T}}(x,z)|_{[\nu]} &\geq \phi'(\mathcal{T}(x,z))(\mathcal{T}_z(x,z) - \frac{1}{\omega(x,z)}|\nabla\mathcal{T}(x,z)|_{[\tilde{\nu}]})\\ &\geq \phi'(\mathcal{T}(x,z))(\mathcal{T}_z(x,z) - \frac{1}{\omega(x,z)}|\nabla\mathcal{T}(x,z)|_{[\nu]}) \geq 0\end{aligned}$$

with $\tilde{\nu} = \frac{\nu}{\phi'(\mathcal{T}(x,z))^2}$ where the second inequality holds because of $|\nabla\mathcal{T}(x,z)|_{[\tilde{\nu}]} \leq |\nabla\mathcal{T}(x,z)|_{[\nu]}$ for $\phi' \geq 1$. Consequently, the resulting TLF satisfies the stability constraint. Moreover, by (5.24) it is strict monotonically increasing w.r.t. z. □

Without considering the perturbed norm, the proof of the above Lemma would work for all monotonically increasing transformation functions with $\phi'(\cdot) > 0$. Thus, the profiles form a convex cone (see Lemma 5.2.2). For the perturbed norm, only transformation functions satisfying (5.24) are feasible.

If the transformation function ϕ is linear (which is meaningful in the view of computational procedures), the rescaled TLF even satisfies the parabolic augmented inequality for any $\varepsilon > 0$ since $\tilde{\mathcal{T}}_{x_i x_i}(x,z) = \phi'(T(x,z))\mathcal{T}_{x_i x_i}(x,z)$. The preceding observation in particular allows for the construction of feasible time labeling functions by rescaling a given one violating the effort constraint. One possible candidate for a rescaling function stems from Proposition 5.2.5. For almost every t there exist

$$r_e(t) = \int\limits_{\Omega} \frac{e(x, p_t(x))}{\mathcal{T}_z(x, p_t(x))} / \kappa(t). \tag{5.25}$$

and the natural choice for the rescaled TLF would be

$$\phi(\mathcal{T}) = r_e(\mathcal{T})\mathcal{T}.$$

Unfortunately, this function is not applicable since the monotonically increasing behavior with respect to t, i.e. $\partial_t r_e(\cdot) \geq 1$, might be violated by. A further possibility, $(1, \sup_t r_e(t))^+$, only works for nondecreasing c which is a strong additional assumption probably not satisfied by the operation. In addition this choice provides, that (5.25) is not differentiable with respect to $\mathcal{T}$.

We base the derivation of the rescaling function on the assumption that the mining operation should always work at highest possible speed, i.e. satisfying $E_{\mathcal{T}}(t) = \mathcal{K}(t)$ for almost all $t \in (0,T)$. The **rescaled time** related to the TLF at time t, i.e. the time a certain profile can be reached under consideration of the effort bound, is given as $\hat{t}(t) = \mathcal{K}^{-1}(E_{\mathcal{T}}(t))$, the solution of

$$E_{\mathcal{T}}(t) = \int_0^{\hat{t}} \kappa(s)ds. \tag{5.26}$$

The time t with the highest exceeding violation of $\hat{t}(t) = t$ determines the rescaling factor.

Lemma 5.2.10. *Consider a time labeling function $\mathcal{T}$ satisfying $\mathcal{T}_z \geq c > 0$ almost everywhere. Then*

$$\mathcal{K}^{-1}(E_{\mathcal{T}}(t))/t$$

is bounded in $[0, T]$ and locally Lipschitz continuous in $(0, T)$

Proof. By (5.26), for any $T \geq t_1 \geq t_2 \geq 0$ we have

$$E_{\mathcal{T}}(t_1) - E_{\mathcal{T}}(t_2) \geq \underline{\kappa}(\hat{t}_1 - \hat{t}_2) \tag{5.27}$$

and

$$t_1 - t_2 = \int_{p_{t_2}(x)}^{p_{t_1}(x)} \mathcal{T}_z(x, z)dz \geq \lambda(p_{t_1}(x) - p_{t_2}(x)).$$

The quantities on both sides of the inequality are nonegative by monotonicity of $\mathcal{T}$. The claimed boundedness now follows for $t_1 = t$, $t_2 = 0$ and combining (5.27) and (5.20), providing

$$\mathcal{K}^{-1}(E_{\mathcal{T}}(t))/t \leq |\Omega|\|e\|_\infty(\underline{\kappa}\lambda)^{-1}.$$

Next consider $t_1 \geq t_2 > 0$. From the estimates above we obtain

$$\begin{aligned}
&|\mathcal{K}^{-1}(E_{\mathcal{T}}(t_1))(t_1)^{-1} - \mathcal{K}^{-1}(E_{\mathcal{T}}(t_2))(t_2)^{-1}| \\
&= |t_2\mathcal{K}^{-1}(E_{\mathcal{T}}(t_1)) - t_1\mathcal{K}^{-1}(E_{\mathcal{T}}(t_1)) + t_1\mathcal{K}^{-1}(E_{\mathcal{T}}(t_1)) - t_1\mathcal{K}^{-1}(E_{\mathcal{T}}(t_2))| \\
&\leq (t_1t_2)^{-1}[\mathcal{K}^{-1}(E_{\mathcal{T}}(t_1))|t_1 - t_2| + t_1|\mathcal{K}^{-1}(E_{\mathcal{T}}(t_1)) - \mathcal{K}^{-1}(E_{\mathcal{T}}(t_2))|] \\
&\leq (t_2)^{-1}(2|\Omega|\|e\|_\infty(\underline{\kappa}\lambda)^{-1})|t_1 - t_2|
\end{aligned}$$

which provides the claimed local Lipschitz continuity. □

The global boundedness of the fraction ensures, that any $\mathcal{T}$ satisfying the underlying (PDE$^\varepsilon$) and having some lower bound on the derivative with respect to z, multiplied by $|\Omega|\|e\|_\infty(\underline{\kappa}\lambda)^{-1}$ is feasible in the sense of the effort constraint. Since this factor is to restrictive, we introduce the rescaled TLF as

$$\phi(\mathcal{T}(x, z)) = \left(1, \sup_{t \in (0,T)} \left\{\frac{\mathcal{K}^{-1}(E_{\mathcal{T}}(t))}{t}\right\}\right)_o^+ \mathcal{T}(x, z)$$

where $(1, \cdot)_o^+$ is a locally smoothed maximum function as depicted in Figure 5.4. The lower bound is necessary because we want to preserve possibly slow behavior of the TLF in certain domains. Here a smoothed version of the pointwise maximum has to be chosen which overestimates the original function like the locally smoothed function from Figure 5.4. The overestimation property is important as underestimating would yield a rescaled time labeling functions not necessarily satisfying the effort constraint as equality.

This construction clearly does not yield a TLF satisfying the boundary condition Ψ but this function was designed to ensure, that any solution of the state equation attains the value

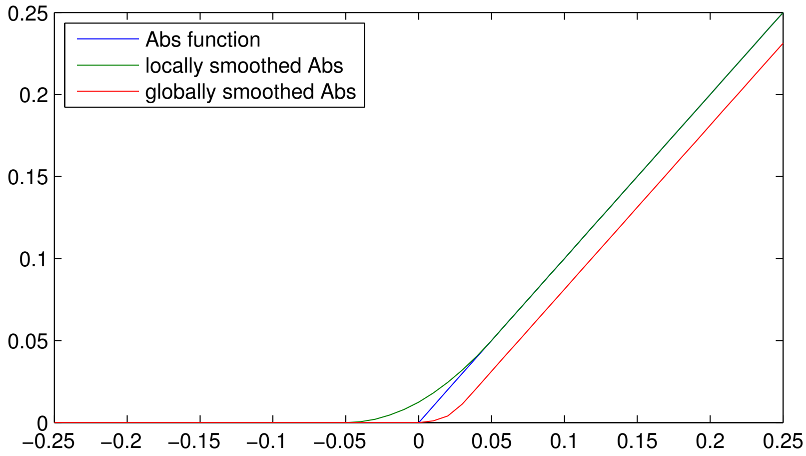


Figure 5.4: Smoothed Max Function for Parameter 0.05

T from Lemma 5.2.5 in a depth of μ for all $x \in \partial\Omega$. By the assumptions, the rescaled TLF satisfies $\mathcal{T}(x,\mu) \geq T$ for all $x \in \partial\Omega$ and thus is still satisfying this requirement.
Note that for the case κ =const the fraction reduces to

$$\frac{\mathcal{K}^{-1}(E_{\mathcal{T}}(t))}{t} = \frac{E_{\mathcal{T}}(t)}{\mathcal{K}(t)}.$$

The section closes with the observation, that $\mathcal{K}^{-1}(E_{\mathcal{T}}(\cdot))$ is continuous in $\mathcal{T}$.

Lemma 5.2.11. *Consider a sequence of time labeling functions $\mathcal{T}_n$ with $\mathcal{T}_n \to \mathcal{T}$ in $C(\mathcal{Q})$. Then $\mathcal{K}^{-1}(E_{\mathcal{T}_n}(\cdot)) \to \mathcal{K}^{-1}(E_{\mathcal{T}}(\cdot))$ in $C([0,T])$.*

Proof. Consider an arbitrary $t \in [0,T]$. By (5.26) the values $\hat{t}_n$ of the function $\mathcal{K}^{-1}(E_{\mathcal{T}_n}(t))$ are defined as the solutions of

$$E_{\mathcal{T}_n}(t) = \int_0^{\hat{t}_n} \kappa(s)ds$$

and for $\mathcal{T}$ we find the value $\hat{t}$ analogously. Since κ is bounded from below we find the estimate

$$\underline{\kappa}|\hat{t}_n - \hat{t}| \leq |E_{\mathcal{T}_n}(t) - E_{\mathcal{T}}(t)|.$$

Further, we obtain from the definition of $E_{\mathcal{T}}(t)$ given in (5.19) the estimate

$$|E_{\mathcal{T}_n}(t) - E_{\mathcal{T}}(t)| \leq \|e\|_{L^\infty(\mathcal{Q})} \int_\Omega |\mathcal{T}_n^{-1}(x)(t) - p_t(x)|dx$$

with $|\mathcal{T}_n^{-1}(x)(t) - p_t(x)| \to 0$ by the uniform convergence of the $\mathcal{T}_n$ to $\mathcal{T}$. Since t was chosen arbitrarily, the estimate holds for the supremum and the claim is proven. $\square$

5.3 The Optimization Problem

In this section we introduce the optimization problem related to the Open Pit Mine Planning. We prove the existence of solutions to the problem on the level of hyperbolic and parabolic operators, show solvability for the parabolic approximating problems and present a corresponding necessary first order optimality condition for the regularized problems. We finish by presenting a convergence result result for solutions of the approximating problems.

5.3.1 Existence of Solutions for the Original Problem

Let

$$\mathcal{V} = \mathcal{Q} \cap hyp(p_u)$$

denote the domain of interest only. In this volume we are able to excavate in a physically stable manner by the properties of the ultimate gain pit. $hyp(p_u) = \{(x,z) : z \leq (p_u)(x)\}$ denotes the **hypograph** of the ultimate pit. The dynamic open pit mine problem is given as

$$\begin{array}{llrcll} \min & -\int\limits_{\mathcal{V}} e^{-\phi(\mathcal{T}(x,z))r} g(x,z) d(x,z) & & & & \\ \text{s.t.} & & u & \in & U_{ad} & \\ & & \mathcal{T} & \in & W^{1,\infty}(\mathcal{Q}) \text{ is viscosity solution of} & \\ & & \mathcal{T}_z + -\frac{1}{\omega}|\nabla\mathcal{T}|_{[\nu]} & = & u & \text{in } \Omega \times [0,\overline{z}] \\ & & \mathcal{T} & = & \Psi & \text{on } \partial\Omega \times [0,\overline{z}] \\ & & \mathcal{T}_z \leq c_z & & & \text{a.e. on } \mathcal{Q} \end{array} \tag{5.28}$$

where U_{ad} was defined in (5.14). In the problem presented above we already fixed the discount function $\varphi(t) = e^{-rt}$. $\phi(t)$ denotes the rescaling function ensuring feasibility with respect to the effort constraint. The upper bound c_z on the partial derivative of $\mathcal{T}$ with respect to depth is introduced to prevent possible jumps of the time labeling function which are naturally introduced by the change of variables (see corresponding discussion in Subsection 5.2.1). Since the value is a technical assumption and can be chosen arbitrary but larger than ν we suggest it to be some power of T, the total time the mining operation needs to excavate $\mathcal{V}$. Then it can be understood as very small fraction of the average speed of the mining operation.
(5.28) is not well posed in the sense of Hadamard (see [51]) since the admissible set $\mathcal{U}_{ad}$ is not closed in $C^{1+\theta}(\mathcal{Q})$. Thus, a sequence of admissible controls may converge to a limit point that is not feasible and hence does not ensure the existence of a viscosity solution for the underlying equation. Consequently, the solutions of the state equation do not depend continuously on the data. In order to guarantee, that a converging sequence of functions in the feasible set admit a limit element in $C^{1+\theta}(\overline{\mathcal{Q}})$, we have to choose the space of control functions such that a bounded sequence of functions contains a subsequence strongly converging in the named Hölder space. In agreement with to Theorem A.2.1 we introduce the set of feasible controls as

$$\mathcal{U} = U_{ad} \cap H^3(\mathcal{Q}).$$

Since $H^3(\mathcal{Q})$ is a Hilbert space, bounded sequences have weakly converging subsequences, which by the compact embedding (see Theorem A.2.1), contain a further subsequence converging strongly in $C^{1+\theta}(\overline{\mathcal{Q}})$. After fixing the function spaces we utilize, as in Chapter 3, a Tikhonov regularization term (see, e.g. [51]) with weight $\beta \in \mathbb{R}^+$ guaranteeing boundedness of an infimizing sequence of admissible controls in $H^3(\mathcal{Q})$.
From the preceding discussion, the rescaling function $\phi(\mathcal{T})$ is defined as essential supremum of $\mathcal{K}^{-1}(E_{\mathcal{T}}(t))/t$. Since the sup norm is nowhere differentiable (see [100]) we have to introduce an additional optimization variable λ to work around this issue. Therefore we introduce the

inequality constraint

$$\mathcal{K}^{-1}(E_{\mathcal{T}}(t))/t \leq \lambda \Leftrightarrow \mathcal{K}^{-1}(E_{\mathcal{T}}(t)) \leq \lambda t.$$

The resulting optimization problem for the Open Pit Mine Planning Problem is finally given as

$$\begin{array}{lrcll}
\min & -\int\limits_{\mathcal{V}} e^{-(1,\lambda)^{+}_{o}\mathcal{T}(x,z)r} g(x,z)d(x,z) + \lambda + \beta\|u\|^2_{H^3} & = & \mathcal{J}(\mathcal{T},u,\lambda) & \\
\text{s.t.} & u & \in & \mathcal{U} & \\
 & \mathcal{T} & \in & W^{1,\infty}(\mathcal{Q}) & \text{is Visc. Sol.} \\
 & \lambda & \in & & \mathbb{R}_+ \\
 & \mathcal{T}_z - \frac{1}{\omega}|\nabla\mathcal{T}|_{[\nu]} & = & u & \text{in } \mathcal{Q} \\
 & \mathcal{T} & = & \Psi & \text{on } \partial\Omega \times [0,\overline{z}] \\
 & \mathcal{K}^{-1}(E_{\mathcal{T}}(t)) & \leq & \lambda t & \text{for a.e. } t \in (0,T) \\
 & \mathcal{T}_z \leq c_z & & & \text{a.e. on } \mathcal{Q}
\end{array} \tag{P_M}$$

Proposition 5.3.1. (P_M) *admits a solution.*

Proof. Let

$$\mathcal{D} = \{(\mathcal{T}(u), u, \lambda)| \text{ triplet satisfies the constraints of } (P_M)\ \}$$

denote the feasible set. Now choose $(\overline{\mathcal{T}}, \overline{u}, \overline{\lambda})$ with

$$\overline{\mathcal{T}}(x,z) = \Psi(z), \overline{u}(x,z) = \Psi_z(z) - \nu/\omega(x,z), \lambda = |\Omega|\|e\|_\infty(\underline{\kappa}\nu)^{-1} + \xi$$

Here, λ is the upper bound on $\mathcal{K}^{-1}(E_{\mathcal{T}}(t))/t$ according to Lemma 5.2.10 increased by a positive constant ξ which is a threshold for strict positivity of $\mathcal{K}^{-1}(E_{\mathcal{T}}(t)) \leq \lambda t$ as equality with the upper bound might hold. Note that the constructed $\overline{\mathcal{T}}$ is a solution for the parabolic problems as well, its gradients are bounded independent of ε and so, $\overline{\mathcal{T}}$ is a viscosity solution of the problem. Consequently, the feasible set $\mathcal{D}$ is nonempty.
Any $\mathcal{T}$ which can be generated by admissible controls is nonnegative, 0 is a global lower bound for all resulting TLF. Choosing $\lambda = 0$ and $u \equiv 0$ we find a lower bound for the objective by

$$-\int\limits_{\mathcal{V}} (g(x,z))^+ d(x,z) \leq -\int\limits_{\mathcal{V}} g(x,z)d(x,z) = \tilde{\mathcal{J}}(0,0,0)$$

although $(0,,0)$ is not feasible. Next we consider an infimizing sequence $\{\mathcal{T}_n, u_n, \lambda_n\}$ with $\mathcal{T}_n = \mathcal{T}(u_n)$ of elements in $\mathcal{D}$ such that

$$\lim_{n\to\infty} \mathcal{J}(\mathcal{T}_n, u_n, \lambda_n) = \inf_{(\mathcal{T},u,\lambda)\in\mathcal{D}} \mathcal{J}(\mathcal{T},u,\lambda) = a$$

Due to the coercivity of the objective, u_n is a bounded sequence in $H^3(\mathcal{Q})$ and thus contains a weakly converging subsequence in $H^3(\mathcal{Q})$ with limit element $\hat{u}$. Since $\mathcal{U} \subset H^3(\mathcal{Q})$ is convex, it is weakly closed and $\hat{u} \in \mathcal{U} \cap H^3$. According to Theorem A.2.1, (along another subsequence denoted the same) u_n converges strongly in $C^{1+\theta}(\overline{\mathcal{Q}})$. For the corresonding viscosity solutions $\mathcal{T}_n = \mathcal{T}(u_n)$ we obtain from the state equation

$$|\nabla\mathcal{T}_n| \leq |\nabla\mathcal{T}_n|_{[\nu]} \leq \omega((\mathcal{T}_n)_z - u_n) \leq \overline{\omega}c_z. \tag{5.29}$$

Thus, they are equicontinuous by the uniform bound on the gradient. Theorem 2.1.1 guarantees the strong convergence in $C(\overline{\mathcal{Q}})$ along a further subsequence to some $\hat{\mathcal{T}}$.

Due to Theorem 4.1.3 and the uniform convergence of u_n we know that the limiting function $\hat{\mathcal{T}}$ is a viscosity solution for $\hat{u}$. Similar to Proposition 5.2.4 we see, that $\hat{\mathcal{T}}$ is Lipschitz continuous and the weak derivatives satisfy the given constraints.

According to Lemma 5.2.11 the function $\mathcal{K}^{-1}(E_{\mathcal{T}}(t))$ is continuous in $\mathcal{T}$. Moreover the sequence $\lambda_n \in \mathbb{R}^+$ contains a convergent subsequence with limit $\hat{\lambda}$ as it is bounded. Passing to the final subsequence we find, that

$$\mathcal{K}^{-1}(E_{\mathcal{T}_n}(t)) \leq \lambda_n t$$

holds for all t and all n. By continuity the same has to be true for the limit $\mathcal{K}^{-1}(E_{\hat{\mathcal{T}}}(t)) \leq \hat{\lambda} t$. Thus, $(\hat{T}, \hat{u}, \hat{\lambda})$ is feasible and we have

$$\mathcal{T}_n \to \hat{\mathcal{T}} \text{ in } C(\overline{\mathcal{Q}}), \quad u_n \rightharpoonup \hat{u} \text{ in } H^3(Q), \quad \lambda_n \to \hat{\lambda} \text{ in } \mathbb{R}$$

The objective is continuous with respect to $\mathcal{T}$ and λ and weakly lower semi continuous with respect to u. Thus we find

$$a = \liminf_{n\to\infty} \mathcal{J}(\mathcal{T}_n, u_n, \lambda_n) \geq \mathcal{J}(\hat{\mathcal{T}}, \hat{u}, \hat{\lambda}) \geq a$$

and the infimum is attained. □

5.3.2 Solutions of the Approximating Optimization Problems

In (P_M) we face two major sources of difficulties. First, the control to state mapping $\mathcal{S}(u)$ of (PDE^0) is only continuous in u and not differentiable. Second, the effort constraint, which is a non local state constraint, is not differentiable when Lipschitz continuous time labeling functions are considered. We have already shown in Proposition 5.2.6, that differentiability of this constraint is obtained for a more regular state function.

Thus, to overcome both drawbacks, we introduce parabolic partial differential equations with an artificial viscosity term, (PDE$^\varepsilon$). For this PDE's existence of solutions and smoothness properties have been studied in Section 5.2.3. While gaining regularity for the states, passing to the parabolic problems introduces other issues. The strict monotonic growth of the resulting TLF with a minimal rate ν is not longer automatically ensured by the state equation due to the term $-\varepsilon\Delta\mathcal{T}$. Moreover, the boundedness of the spatial gradients as derived in (5.29) is no longer given. Consequently, we have to introduce additional inequality constraints by

$$\nu/2 \leq \mathcal{T}_z \quad \text{and } |\nabla\mathcal{T}|_{[\nu]} \leq \alpha.$$

Here the first inequality ensures a minimal growth rate of TLF's. The second one will act like a penalty term for large values of the gradient of $\mathcal{T}$. Again, the quantity α will be introduced as a further optimization variable to the objective. The resulting problem of optimal control subject to a semilinear partial differential equation is given as

$$
\begin{array}{rrcll}
\min & \mathcal{J}(\mathcal{T},u,\lambda)+\alpha & = & & \tilde{\mathcal{J}}(\mathcal{T},u,\lambda,\alpha) \\
\text{s.t.} & u\in\mathcal{U},\ \ \mathcal{T}\in C^{2+\theta,1+\theta/2}(\mathcal{Q}),\ \ \lambda,\alpha & \in & \mathbb{R}_+ & \\
 & \mathcal{T}_z-\varepsilon\Delta\mathcal{T}-\frac{1}{\omega}|\nabla\mathcal{T}|_{[\nu]} & = & u & \text{in } \mathcal{Q} \\
 & \mathcal{T} & = & \Psi & \text{on } \partial\Omega\times[0,\overline{z}] \\
 & \mathcal{K}^{-1}(E_{\mathcal{T}}(t)) & \leq & \lambda t & \text{for a.e. } t\in(0,T) \\
 & \mathcal{T}_z\leq c_z,\ \ \nu/2\leq\mathcal{T}_z,\ \ |\nabla\mathcal{T}|_{[\nu]} & \leq\alpha & & \text{a.e. on } \mathcal{Q}
\end{array}
\tag{5.30}
$$

In this problem constraints on the state $\mathcal{T}$ are still present. Therefore we apply an interior point approach. Note that due to Proposition 2.1.1 and Lemma 2.1.1 the function $|\nabla\mathcal{T}|_{[\nu]}$ is Hölder continuous with constant $\theta<1/2$. Interior Point methods for Hölder continuous functions are well known and have been studied for example in [138, 140].
Let $\mathcal{X}\subset\mathbb{R}^n$ denote an arbitrary n dimensional domain. The barrier functionals, which ensure an abstract pointwise inequality constraint $v(x)\geq 0\ \ \forall x\in\mathcal{X}$ for $v:\mathcal{X}\to\mathbb{R}$ being Hölder continuous with exponent $\tilde{\theta}$, are based on functions $l:\mathbb{R}^+\to\mathbb{R}\cup\{+\infty\}$ defined for any $\sigma\geq 1$ and $\gamma>0$. They are given as

$$
l(s;\gamma,\sigma)=\begin{cases} -\gamma\ln(s) & \text{if } \sigma=1 \\ \frac{\gamma^\sigma}{(\sigma-1)s^{\sigma-1}} & \text{if } \sigma>1 \end{cases}
$$

with derivative

$$
l'(s;\gamma,\sigma)=\frac{-\gamma^\sigma}{s^\sigma}
$$

We define the barrier functionals

$$
b_{\mathcal{X}}(v;\gamma,\sigma)=\int_{\mathcal{X}} l(v(s),\gamma,\sigma)ds
$$

and obtain from [138, Proposition 4.3 and Lemma 7.1], that $v\in C^{\tilde{\theta}}(\mathcal{X})$ and $b_{\mathcal{X}}(v,\gamma,\sigma)<\infty$ for $\sigma\geq n/\tilde{\theta}$ imply $1/v\in C(\mathcal{X})$. Thus, the inequality holds strictly at all points of the domain. Moreover, the directional derivative of $b_{\mathcal{X}}(v,\gamma,\sigma)$ in direction δv is given as

$$
b'(v,\delta v;\gamma,\sigma)=\int_{\mathcal{X}}\frac{-\gamma^\sigma}{v(s)^\sigma}\delta v(s)ds.
$$

Choosing $\sigma=5$, we find, that all pointwise inequality constraints from (5.30) can be expressed by functions $l(s;\gamma,\sigma)$. σ is fixed for the remainder of this section, and we omit it in the following discussion. The complete problem is given as

$$
\begin{array}{ll}
\min & \tilde{\mathcal{J}}(\mathcal{T},u,\lambda) \ \ +b_{\mathcal{Q}}(\alpha-|\nabla\mathcal{T}|_{[\nu]};\gamma)+b_{\mathcal{Q}}(c_z-\mathcal{T}_z;\gamma)+b_{\mathcal{Q}}(\mathcal{T}_z-\nu/2;\gamma) \\
 & \qquad\qquad\quad +b_{(0,\tilde{T})}(\lambda t-\mathcal{K}^{-1}(E_{\mathcal{T}}(t));\gamma)+\alpha \\
\text{s.t.} & \mathcal{T}\in C^{2+\theta,1+\theta/2}(\mathcal{Q}),\ \ u\in\mathcal{U}_{ad},\ \ \lambda\in\mathbb{R}^+,\ \ \alpha\in\mathbb{R}^+ \\
 & \qquad\qquad \mathcal{T}_z-\varepsilon\Delta\mathcal{T}-\frac{1}{\omega}|\nabla\mathcal{T}|_{[\nu]}=u \quad \text{in } \mathcal{Q} \\
 & \qquad\qquad \mathcal{T}=\Psi \quad \text{on } \partial\Omega\times[0,\overline{z}]
\end{array}
\tag{5.31}
$$

Theorem 5.3.1. *For all $\varepsilon>0$ and $\gamma>0$, problem* (5.31) *admits a solution.*

Proof. Let

$$
\mathcal{D}=\{(\mathcal{T}(u),u,M)|\text{ triplet satisfies the constraints of (5.31) }\}
$$

denote the feasible set of (5.31). As shown in Proposition 5.3.1

$$\overline{\mathcal{T}}(x,z) = \Psi(z), \overline{u}(x,z) = \Psi_z(z) - \nu/\omega(x,z), \lambda = |\Omega|\|e\|_\infty(\underline{\kappa}\nu)^{-1} + \xi, \bar{\alpha} = 1$$

are feasible. In particular, the constructed point is inactive with respect to the gradient and effort constraints and the value of the objective is finite. For any solution $\mathcal{T} \in C^{2,1}(\overline{\mathcal{Q}})$ of (PDE$^\varepsilon$) with arbitrary admissible control u the following inequality is satisfied pointwise in $\mathcal{Q}$.

$$\mathcal{T}_z(x,t) - \varepsilon\Delta\mathcal{T}(x,t) \geq 0$$

By the weak maximum principle for parabolic partial differential equations (see Theorem A.3.1), the minimum of $\mathcal{T}$ is attained on the parabolic boundary and thus equals 0. As in Proposition 5.3.1, $\tilde{\mathcal{J}}(\mathcal{T}, u, \lambda)$ can be estimated from below by

$$-\int_{\mathcal{V}} (g(x,z))^+ d(x,z)$$

The partial derivative of $\mathcal{T}$ with respect to z is bounded from above and below simultaneously. Consequently,

$$b_{\mathcal{Q}}(c_z - \mathcal{T}_z; \gamma) + b_{\mathcal{Q}}(\mathcal{T}_z - \nu/2; \gamma)$$

has to be bounded from below. Regarding the constraint on the gradient, we find that

$$\alpha + b_{\mathcal{Q}}(\alpha - |\nabla\mathcal{T}|_{[\nu]}; \gamma)$$

has to be bounded from below and the same holds true for

$$\lambda + b_{(0,T)}(\lambda t - \mathcal{K}^{-1}(E_{\mathcal{T}}(t)); \gamma).$$

Next consider an infimizing sequence $\{\mathcal{T}(u_n), u_n, \lambda_n, \alpha_n\} \in \mathcal{D}$ with

$$\lim_{n\to\infty} \tilde{\mathcal{J}}(\mathcal{T}_n, u_n, \lambda_n, \alpha_n) = \inf_{(\mathcal{T},u,\lambda,\alpha)\in\mathcal{D}} \tilde{\mathcal{J}} = a.$$

Since the barrier functionals are convex (see [138]), the sequence has to be bounded in

$$C^{2+\theta,1+\theta/2}(\mathcal{Q}) \times H^3(\mathcal{Q}) \times \mathbb{R}^+ \times \mathbb{R}^+.$$

Analogously to Proposition 5.3.1, $\{u_n\}$ has a subsequence, strongly converging in $C^{1+\theta}(\mathcal{Q})$. Since $\mathcal{U}$ is weakly closed, the limit u^* is an admissible control. Let $\mathcal{T}^*$ denote the corresonding solution of (PDE$^\varepsilon$). By Lemma 5.2.6, the subsequence of $\mathcal{T}_n$ converges strongly to $\mathcal{T}^*$ in $C^{2,1}(\mathcal{Q})$. For the bounded sequences $\{\lambda_n\}$ and $\{\alpha_n\}$ we obtain the existence of strongly converging subsequences in $\mathbb{R}$ by Bolzano-Weierstrass. Thus we find a subsequence indexed by n satisfying

$$u_n \rightharpoonup u^* \text{ in } H^3, u_n \to u^* \text{ in } C^{1+\theta}(\mathcal{Q}), \mathcal{T}_n \to \mathcal{T}^* \text{ in } C^{2,1}(\mathcal{Q}), \lambda_n \to \lambda^* \text{ and } \alpha_n \to \alpha^* \text{ in } \mathbb{R}^+.$$

By strong convergence of $\mathcal{T}_n$ in $C^{2,1}(\mathcal{Q})$ we find

$$|\nabla\mathcal{T}_n|_{[\nu]} \to |\nabla\mathcal{T}^*|_{[\nu]} \text{ and } \mathcal{K}^{-1}(E_{\mathcal{T}_n}(t)) \to \mathcal{K}^{-1}(E_{\mathcal{T}^*}(t))$$

for all $t \in [0, T]$ implying that $(\mathcal{T}^*, u^*, \lambda^*, \alpha^*)$ is feasible. Moreover, the barrier functionals are continuous in their arguments. So the objective is continuous with respect to $\mathcal{T}_n, u_n$ and α_n and weakly lower semicontinuous with respect to u_n implying

$$a = \liminf_{n\to\infty} \tilde{\mathcal{J}}(\mathcal{T}_n, u_n, \lambda_n, \alpha_n) \geq \tilde{\mathcal{J}}(\mathcal{T}^*, u^*, \lambda^*, \alpha^*) \geq a.$$

Thus the infimum is attained. □

By differentiability properties carried out above, we find the following characterization of optimal solutions to (5.31) by a first order necessary condition (see Theorem 3.1.1).

Theorem 5.3.2. *Let $(\bar{u}, \bar{\lambda}, \bar{\alpha})$ be a local minimizer of (5.31). Then the following inequality has to hold for all directions $(\delta_u, \delta_\lambda, \delta_\alpha)$ with $\delta u = u - \bar{u}$ for all $u \in \mathcal{U}$ and $\delta_\alpha, \delta_\lambda = \pm 1$.*

$$\int_{\mathcal{V}} re^{-(1,\lambda)_o^+ \mathcal{T}(x,z) r} \Big((1,\lambda)_o^+ q(x,z) + (1,\lambda)_o^+)' d_\lambda \mathcal{T}(x,z)\Big) g(x,z) d(x,z) + \beta \langle \bar{u}, \delta_u \rangle_{H^3(\mathcal{Q})}$$
$$+\gamma^5 \int_{\mathcal{Q}} \frac{d_c}{(\alpha - |\nabla \mathcal{T}|_{[\nu]} - \alpha)^5} - \frac{\nabla \mathcal{T} \cdot \nabla q}{(\alpha - |\nabla \mathcal{T}|_{[\nu]})^5 |\nabla \mathcal{T}|_{[\nu]}} dx dt$$
$$+\gamma^5 \int_{\mathcal{Q}} \frac{q_z}{(\mathcal{T}_z - \nu/2)^5} + \frac{q_z}{(c_z - \mathcal{T}_z)^5} dx dt + \int_0^T \frac{\gamma^5 t d_\lambda}{(\lambda t - C^{-1}(E_{\mathcal{T}}(t)))^5} dt + \delta_\alpha$$
$$+ \int_0^T \frac{(\alpha(\mathcal{K}^{-1}(E_{\mathcal{T}}(t))))^{-1}}{(\lambda t - \mathcal{K}^{-1}(E_{\mathcal{T}}(t)))^5} \int_\Omega \frac{e(x, \mathcal{T}^{-1}(t)(x))}{\mathcal{T}_z(x, \mathcal{T}^{-1}(t)(x))} q(x, \mathcal{T}^{-1}(t)(x)) dt \geq 0$$

with

$$\begin{array}{rcll} \mathcal{T}_z - \varepsilon \Delta \mathcal{T} - \frac{1}{\omega} |\nabla \mathcal{T}|_{[\nu]} & = & \bar{u} & \text{in } \mathcal{Q} \\ \mathcal{T} = 0 \text{ on } \Omega \qquad \mathcal{T} & = & \Psi & \text{on } \partial\Omega \times [0, \overline{z}] \\ q_z - \varepsilon \Delta q - \frac{\nabla \bar{\mathcal{T}}}{\omega |\nabla \bar{\mathcal{T}}|_{[\nu]}} \cdot q & = & \delta_u & \text{in } \mathcal{Q} \\ q & = & 0 & \text{on } \Gamma \end{array}$$

After presenting a necessary first order optimality condition we now prove a convergence result for $\varepsilon \to 0$ and $\gamma \to 0$.

Proposition 5.3.2. *Let (ε, γ) be a sequence converging to $(0, 0)$. For any $(\varepsilon, \gamma) > 0$, let $(\mathcal{T}_{\varepsilon,\gamma}, u_{\varepsilon,\gamma}, \lambda_{\varepsilon,\gamma}, c_{\varepsilon,\gamma})$ be a local minimizer of (5.31).*
Then there exist subsequences (ε, γ) and elements $(\mathcal{T}^*, u^*, \lambda^*, \alpha^*) \in W^{1,\infty}(\mathcal{Q}) \times H^3(\mathcal{Q}) \times \mathbb{R}^+ \times \mathbb{R}^+$ such that the following convergence properties are satisfied.

$$\mathcal{T}_{\varepsilon,\gamma} \to \mathcal{T}^* \text{ in } C(\overline{\mathcal{Q}}), u_{\varepsilon,\gamma} \rightharpoonup u^* \text{ in } H^3(\mathcal{Q}), \lambda_{\varepsilon,\gamma} \to \lambda^* \text{ and } c_{\varepsilon,\gamma} \to \tilde{c} \text{ in } \mathbb{R}$$

In addition the point is feasible for (P_M).

Proof. The construction of an element contained in the admissible set of Theorem 5.3.1 is independent of ε and γ. Thus for a positive constant $\xi > 0$

$$\overline{\mathcal{T}}(x,z) = \Psi(z), \overline{u}(x,z) = \Psi_z(z) - \nu/\omega(x,z), \lambda = |\Omega| \|e\|_\infty (\underline{\kappa}\nu)^{-1} + \xi, \bar{\alpha} = 1$$

is feasible for any combination of ε and γ. As already pointed out, the objective is bounded from below and coercive for all γ. Consequently $\{u_{\varepsilon,\gamma}\}$ is bounded and we obtain the existence of a limit element u^* with

$$u_{\varepsilon,\gamma} \rightharpoonup u^* \text{ in } H^3(\mathcal{Q}), \;\; u_{\varepsilon,\gamma} \to u^* \text{ in } C^{1+\theta}(\overline{\mathcal{Q}})$$

along a subsequence. Next we find that $\{\alpha_{\varepsilon,\gamma}\}$ is bounded from above by some $\bar{\alpha}$ and contains a strongly converging subsequence to a limit element α^*. The spatial gradient of each $\mathcal{T}_{\varepsilon,\gamma}$ is bounded by $\bar{\alpha}$ due to the interior point approach. Moreover, the gradient with respect to z is bounded from above and below by c_z and $\nu/2$ respectively independent of ε and γ. Thus, the family $\{\mathcal{T}_{\varepsilon,\gamma}\}$ is equicontinuous and along a further subsequence we find

$$\mathcal{T}_{\varepsilon,\gamma} \to \mathcal{T}^* \text{ in } C(\overline{\mathcal{Q}})$$

with limit element $\mathcal{T}^*$ by Arzela Ascoli. By Proposition 5.2.4 the weak first order derivatives of $\mathcal{T}^*$ are bounded with the same constants as each $\mathcal{T}_{\varepsilon,\gamma}$. Moreover, the limit element satisfies (PDE^0). Thus we can strengthen the bounds of the derivatives to

$$\nu \le \mathcal{T}^*_z \le c_z \quad \text{and } |\nabla \mathcal{T}^*| \le \overline{\omega} c_z.$$

as seen in (5.29). The sequence $\{\lambda_{\varepsilon,\gamma}\}$ is bounded in $\mathbb{R}^+$ and converges, along a subsequence, strongly to some limit element λ^*. Utilizing the strong convergence of $\mathcal{T}_{\varepsilon,\gamma}$, the continuity result Lemma 5.2.11 and the strong convergence of $\lambda_{\varepsilon,\gamma}$, we obtain

$$\lambda^* t - \mathcal{K}^{-1}(E_{\mathcal{T}^*}(t))) \ge 0$$

for all $t \in [0, T]$. Consequently, the element $(\mathcal{T}^*, u^*, \lambda^*)$ is feasible for (P_M). □

The parabolic auxiliary problems do not necessarily yield a TLF satisfying the stability condition but the violation can be quantified. Let $\mathcal{I}(\mathcal{T}) \subset \mathcal{Q}$ denote the set, where a solution $\mathcal{T}$ of (PDE$^\varepsilon$) for arbitrary $\varepsilon > 0$ violates the stability condition. $\mathcal{I}(\mathcal{T})$ is an open set as if the slope constraint is violated at some point $(x, z) \in \mathcal{Q}$ we have

$$\mathcal{T}_z(x,z) - \frac{1}{\omega(x,z)} |\nabla \mathcal{T}(x,z)|_{[\nu]} < 0.$$

By continuity of all involved functions, the same has to hold for all $(\tilde{x}, \tilde{z})$ in a small neighborhood of (x, z). Now we find

$$|\int_{\mathcal{I}(\mathcal{T})} (\mathcal{T}_z - |\nabla \mathcal{T}|_{[\nu]} - u)\varphi| = |\int_{\mathcal{I}(\mathcal{T})} \varepsilon \Delta \mathcal{T} \varphi| \le \varepsilon \bar{\omega} c_z \int_{\mathcal{I}(\mathcal{T})} |\nabla \varphi|$$

for all $\varphi \in C_c^\infty(\mathcal{I}(\mathcal{T}))$.

5.4 Numerical Approximation of the Ultimate Pit

As already discussed in Section 5.2, the ultimate gain pit is of great interest for the open pit mine planning problem as it prescribes the overall time of the mining operation as well as the volume that can be excavated. For a numerical approximation of this object we consider the

Eikonal equation

$$|\nabla p_u(x)| = \omega(x, p_u(x)) \text{ in } \Omega, \;\; p_u(x) = 0 \text{ on } \partial\Omega. \tag{5.32}$$

A viscosity solution can only be expected in unlikely and physically unnatural cases. Thus we compute an approximation of (5.32) based on $\bar{\omega}(x) = \max\{\omega(x,z)|z \in [0,\overline{z}]\}$ and have to solve

$$|\nabla p_u(x)| = \overline{\omega}(x) \text{ in } \Omega, \;\; p_u(x) = 0 \text{ on } \partial\Omega.$$

As mentioned in Section 4.4 there are methods tailored to Eikonal like equations in form of fast marching and fast sweeping methods. We have decided to use the fast sweeping method presented in [130] and described in Algorithm 2, since it is defined for triangular meshes a further numerical consideration of the open pit mine planning problem would require. The base points of the triangularization are given as $\{x_j\}_{j=1}^N$. The procedure employs a Gauss Seidel method to compute iterates $p_{U_j}^{(k)}$ which satisfy the discrete counterpart of the Eikonal equation and converge to the continuous solution for mesh-width tending to zero. For a triangularization without obtuse angles the algorithm is given in Algorithm.

Algorithm 2 Fast Sweeping Method for the Ultimate Gain Pit

DATA: multiple Reference Points x_r^i $(i = 1, ..., R)$, $TOL > 0$, $\{x_j\}$

INITIALIZATION:

- For all reference points $i = 1, ..., R$:
  Sort the nodes $\{x_j\}$ according to the Euclidean norm to the reference points in ascending and descending order and put them in the arrays

  $$\begin{array}{ll} S_i^+ & \text{ascending order for } x_r^i \\ S_i^- & \text{descending order for } x_r^i \end{array}$$

- Assign exact values $p_{U_j}^{(0)} = p_u(x_j)$ for all vertexes on the boundary, keep them fixed during iterations.
- At all other vertexes, assign large positive values to the initial guess $p_{U_j}^{(0)}$

GAUSS SEIDEL: for $k = 0, 1, ...$

  for $i = 1, ..., R$

    for $l = +, -$

      for every vertex C in S_i^l and every triangle associated with C, use local procedure to ensure Eikonal on the triangle by adjusting $p_{U_C}^{(k+1)}$ given the value of p_u on the other vertexes (cf [130] for local solver)

      if $\|p_u^{(k+1)} - p_u^{(k)}\| \leq TOL$, STOP

The following examples are solved on a model domain $\Omega = (1,3) \times (1,3)$ and should give insight into the properties of the ultimate gain pit.
Since we don not consider the dynamic case, the usage of the special boundary treatment is not necessary and we set $\mu = 0$. Note that an incorporation of this concept is straight forward in the algorithm.
The domain is discretized by a uniform mesh of width $h = 1/N$ where N is the number of inner nodes per spatial dimension. We have decided to utilize a single reference point, $x_r = (1.5, 1.5)$. In general, the algorithm terminates if the increment of the Gauss-Seidel

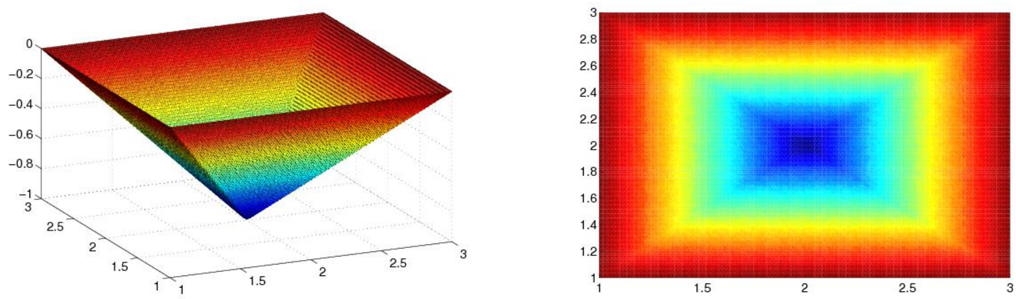
iteration falls below a given tolerance in a certain norm which can be specified by the user. For the presented examples we utilized the Euclidean norm which is sufficient for fixed mesh sizes. If a convergence process of $h \to \infty$ is considered, this has to be replaced by the discrete $W^{1,\infty}(\Omega)$ norm since the solution to (5.32) is an element of this space.

As stopping tolerance we set $TOL = 10^{-8}$. In order to increase readability we considered the range of the profiles to be the negative reals to indicate how deep they reach below the reference level 0. Recalling Example 4.3 we thus compute the viscosity solution to

$$\overline{\omega}(x) - |\nabla p_u(x)| = 0.$$

We have chosen to present solutions of (5.32) for three different functions $\overline{\omega}$.

In Figure 5.5 we depicted the solutions of the algorithm for the choice $\overline{\omega} \equiv 1$. In this case the ore body consist of homogenous material allowing a fixed slope equal to 1 at every point in the volume. The depicted solution reflects the behavior of the block model. In the discrete case a block can only be excavated if the nine blocks above have already been removed. Transferring this cone of dependencies to the continuous setting yields an ultimate pit as in Figure 5.5a. Figure 5.5b illustrates the resulting contours of the pit.



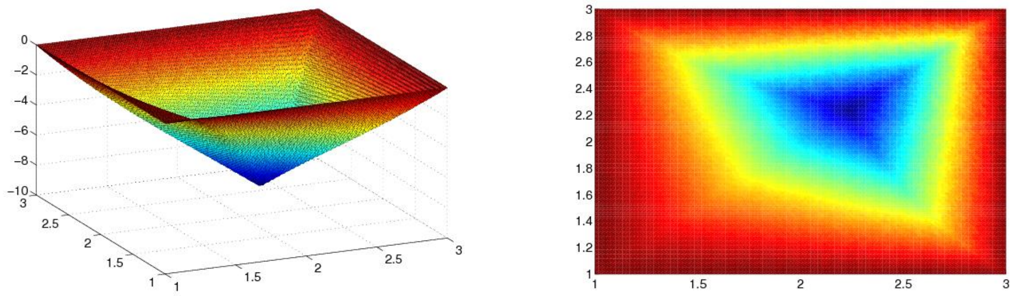
(a) Visualization

(b) Topview

Figure 5.5: Ultimate Gain Pit for $\overline{\omega} \equiv 1$

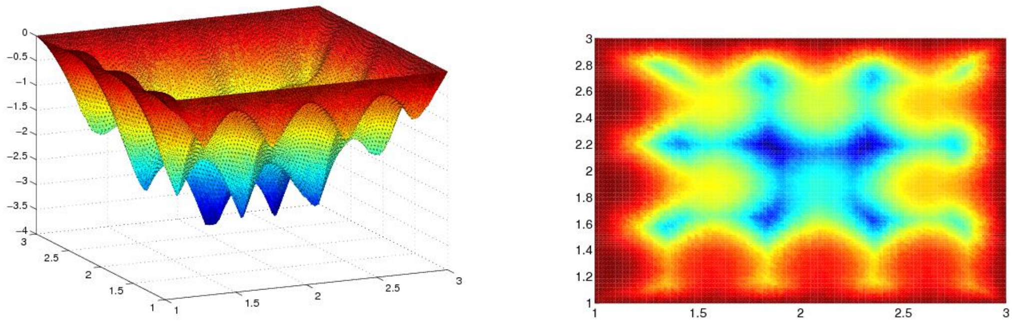
As a second example we have computed p_u for a quadratic right hand side given as $\overline{\omega} = x_1^2 + x_2^2$. Figure 5.6a depicts the result. It represents an ore body where the material hardens in the north eastern direction. This is a more realistic setting for the mine planning problem since it considers a variation of material properties in the ore body. In this instance the profile reaches way deeper than in the preceding example, as can be seen in the corresponding height scale. Again, Figure 5.6b depicts the contours.

Finally we have chosen an academic example where the right hand side of the Eikonal equation is given as the trigonometric function $\overline{\omega} = 5\sin(6x)^2 + 5\sin(5y)^2 + .1$. Although unrealistic in real world applications it demonstrates topological properties of the resulting ultimate pit which is depicted in Figure 5.7a. The form of the right hand side provides several local maximum and minimum points as well as saddle points for the resulting function. This can be seen in the contours shown in Figure 5.7b. The numerical computations were performed on a standard Laptop with 2.50 GHz using Matlab.



(a) Visualization (b) Topview

Figure 5.6: Ultimate Gain Pit for $\overline{\omega} = x_1^2 + x_2^2$



(a) Visualization (b) Topview

Figure 5.7: Ultimate Gain Pit for $\overline{\omega} = 5\sin(6x)^2 + 5\sin(5y)^2 + .1$

5.5 Discussion

Having the ultimate pit at hand, a possible approach for an optimization procedure is to discretize the state equation by the method of lines according to the suggested scheme (4.10). The resulting semidiscretization is a system of ODE's with differentiable right hand side and might be tackled by methods of optimal control of dynamical systems. Here an interpolation scheme has to be used to recover the TLF from the values of the ODE's. We refer to [152] and the references therein for an overview. Unfortunately the effort constraint makes the optimization problem highly problematic in a certain way. Due to this constraint the resulting time labeling function is rescaled by a constant depending on the behavior of the solution of the underlying system at all points of its domain. Consequently, the objective can not be considered in the usual incremental setting which is an integral over the parametrization of an integrand which for fixed parameters only depends on the spatial coordinates. Thus the problem is not of Bolza type and we could not develop a suitable theory to characterize optimal solutions to this problem.
For the fully discretized version of the open pit mine planning problem optimization methods as suggested in [65] might be applicable. Since we put focus on function space results for the problem, we have not considered the fully discrete variant as a piecewise linear problem.

6 Stationary Variational Inequalities with First Order Differential Operators

Introduction

As outlined in Chapter 1, this section deals with transferring some idea of the preceding considerations to the context of optimal control of variational inequalities without time dependence. Here we will in particular utilize the regularization term of the state in the objective.

Consider a given, open domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary for $3 \geq n \geq 1$ and recall the definition of the Hilbert spaces $L^2(\Omega)$, $H_0^1(\Omega)$ and $L_B^2(\Omega)$ from Chapter 2. The optimization problem we will investigate in this section is given as

$$\begin{array}{ll} \inf & \frac{1}{2}|y - y^d|^2 + \frac{\beta}{2}|u|^2 \\ \text{s.t.} & y \in \mathbf{K}, \quad \langle \mathbf{b} \cdot \nabla y + b^0 y - f - u, v - y \rangle \geq 0 \quad \forall v \in \mathbf{K} \end{array}$$

Here the underlying object linking control u and state y is a variational inequality of the first kind with a hyperbolic first order differential operator. We consider a closed convex set of the following form.

$$\mathbf{K} = \{v \in H_0^1(\Omega) \mid v(x) \geq \psi(x) \text{ for almost every } x \in \Omega\}$$

ψ is the so called obstacle and assumed to be an element of $H^2(\Omega)$ Moreover, it has to satisfy $\psi(x) \leq 0$ for all $x \in \partial\Omega$ to ensure that $\mathbf{K}$ is non empty. In the whole section we focus, without loss of generality, on the case $\psi \equiv 0$.

We will discuss the existence of solutions to the underlying variational inequalities

$$\text{find } y \in \mathbf{K} : \langle \mathbf{b} \cdot \nabla y + b^0 y - f, v - y \rangle \geq 0 \text{ for all } v \in \mathbf{K} \tag{VI^0}$$

and therefore introduce a regularized VI by adding a weighted Laplace operator. The resulting elliptic problems

$$\text{find } y_\varepsilon \in \mathbf{K} : \langle -\varepsilon \Delta y_\varepsilon + \mathbf{b} \cdot \nabla y_\varepsilon + b^0 y_\varepsilon - f, v - y_\varepsilon \rangle \geq 0 \text{ for all } v \in \mathbf{K}. \tag{VI^ε}$$

can be treated with existing theory. Afterward, we will analyze the convergence behavior of solutions to (VI^ε) for $\varepsilon \to 0$.

Note, that we utilize a certain kind of elliptic regularization of the differential operator. This should not be mistaken for elliptic regularization as for example in [6] where a parabolic PDE is transformed into an elliptic one by adding a weighted second derivative with respect to time (see [107]).

6.1 Variational Inequalities with First Order Differential Operators

The theory for first order differential operators is often connected to the study of approximating second order ones. Then the results for the original first order operator are established by considering it as degenerate case of a suitable second order object.

Handling the first order part in the differential operators is crucial for many existence and regularity results for general second order variational inequalities. A customary assumption for operators with with first order part is, that they are given in the so called divergence form

$$\nabla \cdot (\mathbf{a}\nabla y) = \sum_{i=1}^{n} a^i y_{x_i x_i} + \sum_{i=1}^{n} a^i_{x_i} y_{x_i}.$$

Here the first order part is incorporated into the second order term. The resulting differential operator is symmetric which yields several desirable properties such as the underlying variational inequality can be seen as characterization of an optimal solution to an optimization problem (see Chapter 3). It directly leads to classical bilevel programming where the feasible set of the optimization problem is described by optimal solutions to a second parametrized optimization problem. This formulation is not applicable to the problems we are going to study in this section as we employ the vanishing viscosity approach for the second order part.

6.1.1 Properties of the First Order Operator

Let the open domain

$$\Omega \subset \mathbb{R}^n$$

with $3 \geq n \geq 1$ be given. We consider a continuously differentiable vector field

$$\mathbf{b} \in C^1(\Omega)^n, \quad \nabla \cdot \mathbf{b} \in C(\Omega).$$

From a physical point of view, $\mathbf{b} \cdot \nabla y$ is an advection term and therefore can be interpreted as transport of some species with concentration y. In addition, we consider a reaction term $b^0 y$ with $b^0 > 0$ almost everywhere. The resulting linear first order differential operator is given as

$$A^0 y = \mathbf{b} \cdot \nabla y + b^0 y \tag{A^0}$$

with domain $D(A^0) \subset L^2(\Omega)$. The formal adjoint of this operator is given as

$$(A^0)^* \varphi = -\mathbf{b} \cdot \nabla \varphi + \tilde{b}^0 \varphi \text{ for all } \varphi \in C_c^\infty(\Omega) \tag{$(A^0)^*$}$$

where $\tilde{b}^0 = b^0 - \nabla \cdot \mathbf{b}$. Note that this in indeed just a formal representation for all test functions known to be dense in any domain space of A^0 we will consider. The closed form of the adjoint operator suffers from the fact, that functions whose image under A^0 is in $L^2(\Omega)$ usually do not have enough regularity for the application of Greens Theorem and obtaining a representation of $(A^0)^*$ by partial integration. Even in the best case, we have to assume additional properties of the differential operator and add the so called coercivity condition which provides a certain kind of coercivity and moreover the uniqueness of solutions to (VI^0).

Definition 6.1.1. (A^0) *fulfills the* ***coercivity condition*** *if*

$$\frac{1}{2}\sum_{i=1}^{n} b^i_{x_i}(x) \leq b^0(x)$$

is satisfied almost everywhere. If the inequality holds strictly, i.e. there exist some $\underline{b}$ *such that*

$$b^0(x) - \frac{1}{2}\sum_{i=1}^{n} b^i_{x_i}(x) \geq \underline{b} > 0 \tag{6.1}$$

holds for almost every $x \in \Omega$*, the operator fulfills the* ***strong coercivity condition****.*

Given some $y \in H^1_0(\Omega)$ we obtain for the advection part of (A^0)

$$\langle \mathbf{b} \cdot \nabla y, y\rangle = (\mathbf{b} \cdot \nabla y, y) = \sum_{i=1}^{n} \left(\int\limits_{\partial\Omega} b^i y^2 ds - \int\limits_{\Omega} y(b^i y_{x_i} + b^i_{x_i} y) \right)$$

and consequently

$$\langle \mathbf{b} \cdot \nabla y, y\rangle = -\left((\frac{1}{2}\sum_{i=1}^{n} b^i_{x_i}) y, y \right).$$

So the coercivity condition allows us to incorporate the first order part into the zero order term without influencing the monotonicity properties of the operator. The condition is well known in the theory of singular operators as pointed out in Chapter 4.

6.1.2 Review of Elliptic Variational Inequalities

As already pointed out, we will obtain solutions to (VI^0) utilizing a vanishing viscosity approach and thus study the behavior of the elliptic variational inequalities (VI^ε). They are defined by the elliptic operators

$$A^\varepsilon y = -\varepsilon \Delta y + \mathbf{b} \cdot \nabla y + b^0 y \tag{A^ε}$$

with the formal adjoints

$$(A^\varepsilon)^* p = -\varepsilon \Delta p - \mathbf{b} \cdot \nabla p + \tilde{b}^0 p. \tag{$(A^\varepsilon)^*$}$$

Again we have $\tilde{b}^0 = b^0 - \sum_{i=1}^n b^i_{x_i}$. The domain space $D(A^\varepsilon)$ is assumed to be $H^1_0(\Omega)$.
The following Lemma collects several useful properties of A^ε.

Lemma 6.1.1. *Let* $\varepsilon > 0$ *be fixed. If the coercivity condition is satisfied, the operator* A^ε *considered as a mapping from* $H^1_0(\Omega)$ *to* $H^{-1}(\Omega)$ *is strongly monotone, bounded, hemicontinuous in the sense of Definition 2.3.1, and coercive.*

Proof. To prove monotonicity we use (6.1), the strong coercivity condition and find

$$\langle A^\varepsilon(y^1 - y^2), y^1 - y^2\rangle \geq \varepsilon \|y^1 - y^2\|^2 + \underline{b}|y^1 - y^2|^2.$$

The other assertions can be verified directly. □

Note that the properties of the preceding result imply pseudo monotonicity of A^ε (see e.g. [105, Proposition 2.5]) as well. In fact it also implies maximal monotonicity (see [8, Theorem II.1.3]) as $H_0^1(\Omega)$ is a reflexive Banach space.
For certain upcoming regularity estimates based on [133] we need the following property.

Lemma 6.1.2. *If the strong coercivity condition is satisfied, $A^\varepsilon\colon L^2(\Omega) \to H^{-2}(\Omega)$ is a* strictly T-Monotone *mapping, i.e.*

$$\langle A^\varepsilon v - A^\varepsilon w, (v-w)^+\rangle > 0 \textit{ for all } v, w \in L^2(\Omega) \textit{ s.t. } 0 \neq (v-w)^+ \in H_0^1(\Omega).$$

Proof. Given $v, w \in L^2(\Omega)$ with $0 \neq (v-w)^+ \in H_0^1(\Omega)$ the linearity of A^ε yields

$$\langle A^\varepsilon v - A^\varepsilon w, (v-w)^+\rangle = \langle A^\varepsilon (v-w)^+, (v-w)^+\rangle + \langle A^\varepsilon (v-w)^-, (v-w)^+\rangle$$

The second term satisfies $\langle A^\varepsilon (v-w)^-, (v-w)^+\rangle = 0$ since $A^\varepsilon(v(x)-w(x))^- = 0$ for almost all $x \in \Omega_+ = \{x \in \Omega | v(x) - w(x) \geq 0\}$ and $\langle A^\varepsilon(v(x)-w(x)), 0\rangle_{H^{-2}(\Omega_+^C), H_0^2(\Omega_+^C)} = 0$. The first term can be estimated as in Lemma 6.1.1 and we obtain

$$\langle A^\varepsilon (v-w)^+, (v-w)^+\rangle \geq \varepsilon \|(v-w)^+\|^2 + \underline{b}|(v-w)^+|^2$$

witch proves the claim. □

For variational inequalities in form of (VI^ε) there is a well developed existence theory. Proposition 2.3.1 is applicable directly and provides the existence of a unique solution $y \in H_0^1(\Omega)$ satisfying the estimate

$$\|(\varepsilon)^{1/2} y\| \leq |f|$$

for all $\varepsilon > 0$. Obviously, given a sequence $\varepsilon \to 0$ and fixing f, the sequence of corresponding solutions has not to be bounded in $H_0^1(\Omega)$. Thus we can not ensure that there exist a weakly converging subsequence of such solutions in the space $H_0^1(\Omega)$ at all and have to impose a further condition providing this property.

Remark 6.1.1. *The preceding result yields a Lipschitz estimate independent of ε for solutions of (VI^ε) in $L^2(\Omega)$ provided the data f are in $L^2(\Omega)$ and the strong coercivity condition holds. This can be seen from the estimate*

$$\varepsilon\|y^1 - y^2\|^2 + \underline{b}|y^1 - y^2|^2 \leq \langle A^\varepsilon(y^1 - y^2), y^1 - y^2\rangle \leq (f^1 - f^2, y^1 - y^2) \leq |f^1 - f^2||y^1 - y^2|$$

which is based on Lemma 6.1.1 and the strong coercivity condition. Now we obtain

$$|y^1 - y^2| \leq (1/\underline{b})|f^1 - f^2| \Rightarrow |y| \leq |f|/\underline{b}$$

and thus a bound in $L^2(\Omega)$ independent of ε.

6.1.3 Solution Concepts for First Order Hyperbolic Variational Inequalities

General Solutions

In this section we present the main existence results for variational inequalities of the first kind with first order differential operators. The result can be found in [135] where the existence of solutions to slightly more difficult variational inequalities was proven. Instead of mixed

Neumann and Dirichlet boundary conditions as in the reference we only focus on Dirichlet conditions on the whole boundary. The existence theorem according to Rodrigues is given as follows.

Theorem 6.1.1. *Let $f \in L^2(\Omega)$ be given. Consider a sequence of viscosity parameters $\varepsilon \to 0$ and the corresponding unique solutions solutions y_ε of (VI^ε). Then there exist a unique $y \in L^2_B(\Omega)$ such that*

$$y_\varepsilon \to y \text{ in } L^2(\Omega)$$

and y solves

$$\text{find } y \in \overline{\mathbf{K}}^{L^2_B(\Omega)} : \langle \mathbf{b} \cdot \nabla y + b^0 y - f, v - y \rangle \geq 0 \text{ for all } v \in \mathbf{K}. \tag{VI^0_B}$$

In addition y satisfies the so called first order Lewy-Stampacchia inequality

$$f \leq A^0 y \leq (f)^+.$$

The proof is completely analogous to the one given in [135] despite the discussion of the boundary conditions which significantly simplifies in our setting. Note, that by density we could extend the variational inequality to all functions $v \in \overline{\mathbf{K}}^{L^2(\Omega)}$. The uniqueness proof is briefly sketched in the following Lemma.

Proposition 6.1.1. *Let A^0 satisfy the strong coercivity condition. Then the solution is unique in $L^2_B(\Omega)$.*

Proof. Let $y^1, y^2 \in L^2_B(\Omega)$ denote two solutions of (VI^0_B) for one particular $f \in L^2(\Omega)$. As mentioned, the variational inequality can be extended to all $v \in \overline{\mathbf{K}}^{L^2(\Omega)}$ by density. Thus y^1 and y^2 are feasible test functions and using them as argument in the corresponding variational inequalities yields

$$0 \geq \langle A^0(y^1 - y^2), y^1 - y^2 \rangle \geq \underline{b}|y^1 - y^2|^2$$

for the sum of both according to the strict coercivity assumption. Here we use that Greens formula is valid for $v \in L^2_B(\Omega)$ if $A^0 v$ is paired with v (otherwise only for pairings with $\varphi \in H^1(\Omega)$, see [135]). Thus, $y^1 = y^2$ in $L^2(\Omega)$ implying $y^1(x) = y^2(x)$ almost everywhere. Thus $y^1 = y^2$ in $L^2_B(\Omega)$. □

Since $H^1_0(\Omega) \subset L^2_B(\Omega)$, the result also holds for more regular solutions. In general the space $L^2_B(\Omega)$ is larger than $H^1(\Omega)$. Even in the case $\Omega \subset \mathbb{R}^1$ $L^2_B(\Omega)$ need not be equal to $H^1(\Omega)$ as by Hölder inequality we can only estimate

$$|b^1 y_x| \leq \|b^1\|_{L^\infty(\Omega)} |y_x|$$

where the left side is finite for $y \in L^2_B(\Omega)$ but this need not be true for the right hand side. In several space dimensions the weak derivatives of y might share singularities providing, that the sum of them is an element of $L^2(\Omega)$, while this fails for the weak derivatives alone. We finish this part by noticing, that the set of solutions to (VI^0_B) is not weakly closed. Consider a bounded and weakly converging sequence of data $\{f_n\}$ with corresponding solutions $\{y_n\}$ in $L^2_B(\Omega)$ of (VI^0_B). Similar to Remark 6.1.1 we find

$$|y_n| \leq c|f_n| \leq \tilde{c}$$

providing an upper bound on the sequence of solutions in $L^2(\Omega)$ by. Thus we can find a further subsequence indexed by n such that $\{(y_n, f_n)\}$ converges weakly in $L^2(\Omega) \times L^2(\Omega)$ with limit elements (y, f) in this spaces. This is not sufficient to obtain

$$\lim_{n\to\infty}(\mathbf{b}\cdot\nabla y_n + b^0 y_n - f_n, v - y_n) = (\mathbf{b}\cdot\nabla y + b^0 y - f, v - y)$$

since here we need strong convergence of y_n in $L^2(\Omega)$.

Solutions in the Viscosity Sense

In this section we discuss the concept of solutions in the viscosity sense. First we observe a basic property of the operator A^0 for a certain domain space.

Lemma 6.1.3. *A^0 is a bounded mapping from $H_0^1(\Omega)$ to $L^2(\Omega)$.*

Proof. We have

$$\sup_{\|\varphi\|=1} |A^0\varphi| \leq \sup_{\|\varphi\|=1} \sum_{i=1}^{n} |b^i|_{L^\infty(\Omega)}|\varphi_{x_i}| + |b^0|_{L^\infty(\Omega)}|\varphi| \leq c$$

$\square$

By Definition 3.2.1 and $L^2(\Omega)^* = L^2(\Omega)$, the adjoint mapping $(A^0)^*$ is well defined for $D(A^0) = H_0^1(\Omega)$. Next we will define the mathematical object of solutions in the viscosity sense of first or hyperbolic variational inequalities.

Definition 6.1.2. *Consider the problem*

$$\text{find } y \in \mathbf{K} : \langle \mathbf{b}\cdot\nabla y + b^0 y - f, v - y\rangle \geq 0 \text{ for all } v \in \mathbf{K}$$

for a closed convex set $\mathbf{K} \subset H_0^1(\Omega)$ and given data $f \in L^2(\Omega)$.
*$y \in H_0^1(\Omega)$ is called **solution in the viscosity sense** if there exist a sequence of parameters $\varepsilon \to 0$ and corresponding pairings $(y_\varepsilon, f_\varepsilon)$ satisfying*

$$\text{find } y_\varepsilon \in \mathbf{K} : \langle -\varepsilon\Delta y_\varepsilon + \mathbf{b}\cdot\nabla y_\varepsilon + b^0 y_\varepsilon - f, v - y_\varepsilon\rangle \geq 0 \text{ for all } v \in \mathbf{K}$$

and

$$y_\varepsilon \rightharpoonup y \text{ in } H_0^1(\Omega) \quad \text{and } f_\varepsilon \rightharpoonup f \text{ in } L^2(\Omega).$$

Solutions in the viscosity sense are designed, such that they can be approximated by solutions of elliptic variational inequalities in the weak sense. Moreover, the solution of (VI^0) is of the same regularity as the solutions of the approximating variational inequalities (VI^ε). So far, the introduced concept of solutions is artificial since it mainly introduces a strong restriction on the data f. However, with

$$y \equiv 0 \quad \text{and} \quad f \equiv 0$$

and approximating solutions

$$y_\varepsilon \equiv 0 \quad \text{and} \quad f_\varepsilon \equiv 0 \quad \text{for all } \varepsilon > 0$$

the set of solutions of this type is nonempty. To prove the existence of further solutions of this kind we introduce the following auxiliary result.

Lemma 6.1.4. (VI^0) *is equivalent to the complementarity system*

$$\mathbf{b} \cdot \nabla y + b^0 y - f = \xi \quad \xi \geq 0 \quad y \geq 0 \quad (\xi, y) = 0$$

Proof. The proof is similar to e.g. [7, Theorem 2.5]. A solution of (VI^0) satisfies $y \geq 0$ by definition. By the claimed regularity of y, f and u, we naturally obtain $\xi \in L^2(\Omega)$. Thus we find

$$\langle \xi, v - y \rangle = \int_\Omega \xi(v - y) \geq 0$$

for all $v \in \mathbf{K}$. By density, we can extend this inequality to all $v \in \{\phi \in L^2(\Omega) | \phi(x) \geq 0 a.e.\}$. Now testing with $v = y + \eta, \eta \in L^2(\Omega)^+$ arbitrary, provides $\xi \geq 0$ a.e.. For $v = 0$ and $v = 2y$ we in addition obtain $(\xi, y) = 0$.
The converse direction follows directly from the fact, that $v \geq 0$ almost everywhere for all $v \in \mathbf{K}$ □

With this result, pairings $(y, f) \in H_0^1(\Omega) \times L^2(\Omega)$ in the sense of Definition 6.1.2 can be constructed as in the following example.

Example 6.1. *Consider the domain $\Omega = (-2, 2) \subset \mathbb{R}^1$. Let the state $y = y_\varepsilon$ be given as the smooth function*

$$y(x) = e^{-(1-x^2)^{-1}} \chi_{(-1,1)}$$

with the corresponding first and second derivatives. For $\mathbf{b} = 1$ and $b^0 = c > 0$ we construct

$$f_\varepsilon(x) = (-\varepsilon \Delta y + \mathbf{b} \cdot \nabla y + cy)\chi_{(-1,1)}$$

and obtain $\xi_\varepsilon = 0$. Moreover, $f_\varepsilon \to f = \mathbf{b} \cdot \nabla y + cy$ in the sup norm and consequently in $L^2(\Omega)$. So $(y, A^0 y)$ is a solution in the viscosity sense and clearly not equal to zero. For several spatial dimensions we use a rotation of the bump function and obtain a similar result.

The preceding example provides a methodology such that for any $y \in C_c^2(\Omega)$ with $y \geq 0$ for all $x \in \Omega$ we can construct corresponding data f such that $\xi(x) \geq 0$ for almost every $x \in \Omega$ and $(y, \xi) = 0$. The procedure can easily adapted to non-trivial obstacles.
For the remainder of this section we split up the data in some fixed distributed load $f \in L^2(\Omega)$ and some variable part $u \in L^2(\Omega)$ which will be referred to as control. The following result proves that any pairing $(y, u) \in H_0^1(\Omega) \times L^2(\Omega)$ satisfying (VI^0) for given data $f \in L^2(\Omega)$ is a solution of (VI^0) in the viscosity sense. This is shown by the application of a singularity argument as presented in [133].

Theorem 6.1.2. *Let (A^0) satisfy the strong coercivity condition and consider a pairing $(y, u) \in H_0^1(\Omega) \times L^2(\Omega)$ satisfying the variational inequality*

$$y \in \mathbf{K} : \langle \mathbf{b} \cdot \nabla y + b^0 y - f - u, v - y \rangle \geq 0 \;\; \forall \;\; v \in \mathbf{K}. \tag{6.2}$$

Moreover, consider a sequence of parameters $\varepsilon \to 0$ and an arbitrary shift $h \in L^2(\Omega)$.
Then there exist solutions to (VI^ε) for data $f + u + \varepsilon h$ such that

$$y_\varepsilon \to y \text{ in } H_0^1(\Omega) \quad \text{and} \quad u_\varepsilon \to u \text{ in } L^2(\Omega)$$

along a subsequence.

Proof. The strong convergence $u_\varepsilon \to u$ in $L^2(\Omega)$ holds by construction of the shift. Given $u_\varepsilon \in L^2(\Omega)$, for any $\varepsilon > 0$ the existence of a unique solution y_ε to (VI^ε) is ensured by Proposition 2.3.1. The claimed regularity of y allows to test (VI^ε) with it for all $\varepsilon > 0$. Testing (6.2) with y_ε and adding the inequalities provides

$$\langle -\varepsilon\Delta y_\varepsilon + \mathbf{b}\cdot\nabla(y_\varepsilon - y) + b^0(y_\varepsilon - y) - \varepsilon h, y - y_\varepsilon\rangle \geq 0.$$

Adding and subtracting $\varepsilon\Delta y$ we find by Young's inequality

$$\begin{aligned}\langle -\varepsilon\Delta(y - y_\varepsilon) + \mathbf{b}\cdot\nabla(y - y_\varepsilon) + b^0(y - y_\varepsilon), y - y_\varepsilon\rangle &\leq \varepsilon\langle -\Delta y - h, y - y_\varepsilon\rangle\\ &\leq \varepsilon(\|y\| + |h|)\|y - y_\varepsilon\|\\ &\leq \varepsilon/2(\|y\| + |h|)^2 + \varepsilon/2\|y - y_\varepsilon\|^2\end{aligned}$$

Utilizing the strong coercivity condition we can estimate

$$0 \leq \varepsilon\|y - y_\varepsilon\|^2 + \underline{b}|y - y_\varepsilon|^2 \leq \langle -\varepsilon\Delta(y - y_\varepsilon) + \mathbf{b}\cdot\nabla(y - y_\varepsilon) + b^0(y - y_\varepsilon), y - y_\varepsilon\rangle$$

and obtain

$$\begin{aligned}\|y - y_\varepsilon\|^2 &\leq (\|y\| + |h|)^2\\ |y - y_\varepsilon|^2 &\leq \varepsilon/(2\underline{b})(\|y\| + |h|)\\ \varepsilon\|y - y_\varepsilon\|^2 &\leq \varepsilon\langle -\Delta y - h, y - y_\varepsilon\rangle\end{aligned} \tag{6.3}$$

The first estimate provides boundedness of y_ε in $H_0^1(\Omega)$. Thus there exists a $w \in H_0^1(\Omega)$ such that

$$y_\varepsilon \rightharpoonup w \text{ in } H_0^1(\Omega)$$

along a subsequence. By the second inequality we obtain strong convergence $y_\varepsilon \to y$ in $L^2(\Omega)$. Consequently $w = y$ has to hold for the weak limit of the previous weak convergence in $H_0^1(\Omega)$ since

$$\lim_{\varepsilon\to 0}\langle \phi, y_\varepsilon\rangle = \langle \phi, y\rangle \forall \phi \in C_c^\infty(\Omega)$$

and the density of this space in $H^{-1}(\Omega)$. The third estimate yields

$$0 \leq \limsup_{\varepsilon\to 0}\|y - y_\varepsilon\|^2 \leq \lim_{\varepsilon\to 0}\langle -\Delta y - h, y - y_\varepsilon\rangle = 0$$

implying strong convergence in $H_0^1(\Omega)$. □

Theorem 6.1.2 essentially relies on the strong coercivity condition. Without this condition the strong convergence $y_\varepsilon \to y$ in $L^2(\Omega)$ can not be established and thus we can not obtain $y_\varepsilon \rightharpoonup y$ in $H_0^1(\Omega)$ and the consequences from this property.
This implies that we can only consider operators A^0 acting linear on the first order weak derivatives of the states. For nonlinear operators, estimates like (6.3) are not valid.
The preceding result proves, that any solution of the first order variational inequality with a particular regularity can automatically be approximated by solutions to elliptic problems. Thus we can drop the artificial term solution in the viscosity sense which only reflects this approximation property. For the remainder of this chapter we will only talk about (regular) solutions.

We point out, that the set of parings solving the variational inequality problem is non empty. This follows from the fact, that the constraint set $\mathbf{K}$ is non empty by construction. Fixing an arbitrary $y \in \mathbf{K}$ and setting $u = A^0 y + f$, clearly ensuring $u \in L^2(\Omega)$, (VI^0) is satisfied and by Theorem 6.1.2 we have a solution in the sense of Definition 6.1.2.
The following example will demonstrate the existence of feasible pairings $(y, u) \in H_0^1(\Omega) \times L^2(\Omega)$ for the first order variational inequality problem. In addition it provides a counter example for the claim, that the problem can be reduced control functions u of the form

$$A^0 w + f \text{ for some } w \in H_0^1(\Omega). \tag{6.4}$$

Example 6.2. *Consider the setting $\Omega \subset \mathbb{R}^n$, $n = 1, 2$ and $f \equiv 0$. We further assume the existence of open balls $B_r(\hat{x}) \subset \Omega$ and $B_{r/2}(\hat{x}) \subset B_r(\hat{x})$ with $\hat{x} \in \Omega$ and radius $r > 0$. Now we define the differential operator as follows. Let the components of $\mathbf{b}$ be given as*

$$\mathbf{b^i}(x) = \begin{cases} 1 & x \in \Omega \backslash B_r(\hat{x}) \\ 0 & x \in B_{r/2}(\hat{x}) \end{cases}$$

and let there be a smooth transition of $\mathbf{b^i}(x)$ from 1 to 0 in $B_r(\hat{x}) \cap B_{r/2}(\hat{x})$. Finally, set $b^0 = \alpha$ with $\alpha > 0$ large enough such that the strong coercivity condition (Definition 6.1.1) is satisfied. Next, we fix a function $y \in H_0^1(\Omega)$ with an active set Ω_0 that contains $B_r(\hat{x})$. These requirements are met for example by choosing the obstacle $y = \psi \equiv 0$. Setting

$$u = \begin{cases} A^0 y & x \in \Omega \backslash B_{r/2}(\hat{x}) \\ \tilde{u} & x \in B_{r/2}(\hat{x}) \end{cases}$$

where $\tilde{u}$ is an arbitrary non-positive function in $L^2(B_{r/2}(\hat{x}))$, the variational inequality problem (VI^0) provides for an arbitrary $v \in \mathbf{K}$

$$\begin{aligned} \langle A^0 y - u, v - y \rangle &= (A^0 y - u, v - y)_{L^2(\Omega \backslash B_{r/2}(\hat{x}))} + (A^0 y - u, v - y)_{L^2(B_{r/2}(\hat{x}))} \\ &= -(\tilde{u}, v)_{L^2(B_{r/2}(\hat{x}))} \geq 0. \end{aligned}$$

Thus, (y, u) satisfies the variational inequality problem. To show, that u not necessarily has the representation (6.4), we recall that $W_0^{1,1}(B_{r/2}(\hat{x}))$ embeds continuously into $L^2(B_{r/2}(\hat{x}))$ (see, e.g. [2]) and we consequently find $A^0 u = \alpha u \in L^2(B_{r/2}(\hat{x}))$ for any function $u \in W_0^{1,1}(B_{r/2}(\hat{x}))$. Since $W_0^{1,1}(B_{r/2}(\hat{x}))$ is larger than $W_0^{1,2}(B_{r/2}(\hat{x}))$, the claim follows. Moreover, we have a solution in the sense of Definition 6.1.2 according to Theorem 6.1.2.

Example 6.2 utilizes, that the differential operator is degenerated in that the vector field $\mathbf{b}$, acting on the first order weak derivatives of y, vanish on an open part of the domain. So, although showing that the structure of feasible functions u is not restricted to the representation (6.4) for the entire domain Ω, the construction provides, that u, restricted to $\Omega \backslash \overline{B_{r/2}(\hat{x})}$, equals $A^0 w$, where w is an element of $H_0^1(\Omega \backslash \overline{B_{r/2}(\hat{x})})$. Consequently, u admits the claimed representation on $\Omega \backslash \overline{B_{r/2}(\hat{x})}$.

Discussion

We point out, that general solutions in $L_B^2(\Omega)$ and solutions in $H_0^1(\Omega)$ have similar properties as the solutions of first order partial differential equations, entropy and viscosity solutions as introduced in Chapter 4, in a certain sense. Both are derived as limits of auxiliary problems where the convergence process is considered in different norms. In the second, more regular case the solutions are obtained whenever the convergence is in $H^1(\Omega)$, i.e. the gradients of the solutions to the auxiliary problems are bounded independent of the viscosity parameter.

6.1.4 Formulation of the Problem and Existence of Solutions

We have already pointed out that the feasible set of the prototype problem

$$\begin{array}{lrl} \min & \frac{1}{2}|y-y^d|^2+\frac{\beta}{2}|u|^2 & =\mathcal{J}(u,y) \\ \text{s.t.} & y\in\mathbf{K},\quad \langle \mathbf{b}\cdot\nabla y+b^0y-f-u,v-y\rangle\geq 0 & \forall v\in\mathbf{K} \end{array}$$

is not weakly closed in $L^2_B(\Omega)$. Thus we have to use a regularization to obtain a solvable problem. This can be done in two different ways. First we may only regularize the differential operator with a viscosity term and obtain for every $\varepsilon > 0$ a problem of optimal control subject to an elliptic variational inequality. For this case, theoretical and numerical results has been established by several authors started with the pioneering work by Mignot and Puel [119], further developed by Hintermüller and Kopacka [73] and then considered for example in [71, 79, 115].

This regularization changes the problem significantly since we loose the first order nature of the differential operator. Therefore we concentrate on regularization approaches preserving the nature of the operator.

In Section 6.1.3 we demonstrated, that the lack of closedness of the feasible set is caused by the possible weak-weak convergence of an infimizing sequence (y_n, u_n) in $L^2(\Omega)\times L^2(\Omega)$ which does not allow for the conclusion

$$\lim_{n\to\infty}(y_n,u_n)=(\tilde{y},\tilde{u})$$

where $(\tilde{y},\tilde{u})$ are the corresponding weak limits. According to [157], any regularization technique ensuring either y_n or u_n to converge strongly in $L^2(\Omega)$ is sufficient for the desired property. The compact embedding

$$H^1_0(\Omega)\to L^2(\Omega)$$

provides strong convergence along a subsequence in $L^2(\Omega)$ if the original sequence converges weakly in $H^1_0(\Omega)$. So a Tikhonov regularization of either u or y for the space $H^1_0(\Omega)$ is sufficient. Both cases are restrictive constraints on feasible control functions u since in the first case only very regular functions are allowed while the second case every feasible control has to be chosen such that the regularity of the corresponding state is satisfied.

Since the second approach allows less regular functions u we focus on this case and derive stationarity conditions directly on the hyperbolic level.

Moreover, we will use the vanishing viscosity approach as an additional regularization, use existing theory to obtain a stationarity system for the resulting problem and study its behavior, when the viscosity parameter is driven to zero.

The Hyperbolic Problem

Consider

$$\begin{array}{lr} \min & \frac{1}{2}|y-y^d|^2 + \frac{\tilde{\beta}}{2}\|y\|^2 + \frac{\beta}{2}|u|^2 \quad = \mathcal{J}(u,y) \\ \text{s.t.} \quad y \in \mathbf{K}, & \langle \mathbf{b}\cdot\nabla y + b^0 y - f - u, v - y\rangle \geq 0 \quad \forall v \in \mathbf{K} \end{array} \qquad (\tilde{P})$$

The feasible set is weakly closed and allows us to establish the following result. The proof uses standard arguments of optimization theory in separable Banach spaces.

Proposition 6.1.2. $(\tilde{P})$ *admits a solution* $(y,u) \in L^2_B(\Omega)\cap H^1_0(\Omega)\times L^2(\Omega)$.

Proof. Let

$$\mathcal{D} = \{(y,u)|(y,u) \text{ satisfies } y\in\mathbf{K},\ \ \langle \mathbf{b}\cdot\nabla y + b^0 y - f - u, v-y\rangle \geq 0 \ \ \forall v\in\mathbf{K}\}$$

denote the feasible set. For $y\equiv 0$ and $u=-f$ we find by Lemma 6.1.1 a feasible point. Next consider an infimizing sequence $(y_n,u_n)\in\mathcal{D}$ with

$$\lim_{n\to\infty}\mathcal{J}(y_n,u_n) = \inf_{(y,u)\in\mathcal{D}}\mathcal{J}(y,u) = a.$$

The objective functional $\mathcal{J}$ is bounded from below and coercive. Due to the feasible point, $\|y_n\|$ and $|u_n|$ are bounded by $\min\{\tilde{\beta},\beta\}^{-1/2}|y^d|$. Consequently, there exist weakly converging subsequences (denoted the same) and corresponding limit elements $\tilde{y}\in H^1_0(\Omega)$ and $\tilde{u}\in L^2(\Omega)$ with

$$y_n \rightharpoonup \tilde{y} \text{ in } H^1_0(\Omega)\ \ , y_n\to\tilde{y} \text{ in } L^2(\Omega) \quad \text{and} \quad u_n\rightharpoonup\tilde{u} \text{ in } L^2(\Omega).$$

Since $\mathbf{K}$ is convex and closed it is weakly closed (see Theorem A.4.1) implying $\tilde{y}\in\mathbf{K}$. Moreover, $|\mathbf{b}\cdot\nabla y_n|$ is bounded due to the Cauchy-Schwarz inequality and the regularity of y and thus contains a weakly converging subsequence (denoted the same) with

$$\mathbf{b}\cdot\nabla y_n \rightharpoonup \mathbf{b}\cdot\nabla\tilde{y} \text{ in } L^2(\Omega).$$

Now the final subsequence provides

$$0 \leq \lim_{n\to\infty}(\mathbf{b}\cdot\nabla y_n + b^0 y_n - f - u_n, v - y_n) = (\mathbf{b}\cdot\nabla\tilde{y} + b^0\tilde{y} - f - \tilde{u}, v-\tilde{y})$$

for any $v\in\mathbf{K}$. Thus the limit elements $(\tilde{y},\tilde{u})\in\mathcal{D}$ are feasible. The weak lower semicontinuity of the objective yields

$$a \leq \mathcal{J}(\tilde{y},\tilde{u}) \leq \liminf_{n\to\infty}\mathcal{J}(y_n,u_n) = a$$

and the infimum is attained. □

Note that considering Lemma 6.1.4, the preceding result also guarantees the existence of a solution of

$$\begin{array}{lr} \min & \mathcal{J}(u,y) \\ \text{s.t.} & \mathbf{b}\cdot\nabla y + b^0 y - f - u = \xi, \xi\geq 0, y\geq 0, (\xi,y)=0 \end{array} \qquad (P)$$

The Vanishing Viscosity Technique

In addition to the argumentation on the hyperbolic level we consider a second stage of regularization where we introduce a second order term to the differential operator whose influence will be decreased in a further convergence process. As a consequence we obtain a problem of optimal control subject to an elliptic variational inequality as studied in [119, 139, 73]. The main idea is, that we will obtain stronger stationarity condition for this case and study, whether they can be preserved for the weight tending to zero. We introduce the regularized problem for $(\tilde{P})$ as

$$\begin{array}{lll} \min & \mathcal{J}(u,y) = \frac{1}{2}|y-y^d|^2 + \frac{\tilde{\beta}}{2}\|y\|^2 + \frac{\beta}{2}|u|^2 & \\ \text{s.t.} & \qquad\qquad\qquad\qquad\qquad\qquad y & \in \mathbf{K} \\ & \langle -\varepsilon\Delta y + \mathbf{b}\cdot\nabla y + b^0 y - f - u, v - y\rangle \quad \geq 0 & \forall v \in \mathbf{K} \end{array} \qquad (\tilde{P}^\varepsilon)$$

For the following results we fix $\varepsilon > 0$. Similar to Lemma 6.1.4 the following result is proven and can be found in [7].

Lemma 6.1.5. *The VI of the first kind (VI^ε) is equivalent to the complementarity system*

$$-\varepsilon\Delta y + \mathbf{b}\cdot\nabla y + b^0 y - f - u = \xi, \quad \xi \geq 0 \quad y \geq 0 \quad (\xi, y) = 0 \qquad (6.5)$$

Note, that any solution to an elliptic VI is uniquely characterized by the weaker complementarity system

$$\begin{gathered} -\varepsilon\Delta y + \mathbf{b}\cdot\nabla y + b^0 y - f - u = \xi \text{ in } H^{-1}(\Omega), \\ \langle \xi, v\rangle \geq 0 \text{ for all } v \in H_0^1(\Omega)^+ \quad y \geq 0 \quad \langle \xi, y\rangle = 0 \end{gathered}$$

with $H_0^1(\Omega)^+ = \{v \in H_0^1(\Omega) | v \geq 0 \text{ a.e.}\}$. The pointwise interpretation of ξ as an $L^2(\Omega)$ function in (6.5) is the result of regularity theory for solutions of elliptic variational inequalities as in [133, chapter 5].
Due to Lemma 6.1.5 we can present an equivalent formulation of $(\tilde{P}^\varepsilon)$, namely

$$\begin{array}{lrl} \min & \mathcal{J}(u,y) = \frac{1}{2}|y-y^d|^2 + \frac{\tilde{\beta}}{2}\|y\|^2 + \frac{\beta}{2}|u|^2 & \\ \text{s.t.} & -\varepsilon\Delta y + \mathbf{b}\cdot\nabla y + b^0 y - f - u & = \xi \\ & \xi \geq 0, y \geq 0, (\xi, y) & = 0 \end{array} \qquad (P^\varepsilon)$$

for which the next result can be found in [7, 15, 94].

Proposition 6.1.3. $(\tilde{P}^\varepsilon)$*, and thus* (P^ε)*, admits a solution.*

Next we proof the consistency of $(\tilde{P}^\varepsilon)$ with $(\tilde{P})$. This is carried out in two steps. First we show, that a sequence of solutions to the elliptic problems weakly converges to a feasible point of $(\tilde{P})$ along a subsequence.

Proposition 6.1.4. *Consider a sequence of viscosity parameter $\varepsilon \to 0$. and let $(y_\varepsilon, u_\varepsilon)$ denote solutions to the problems (P^ε). Then for a subsequence (again indexed by ε) $(y_\varepsilon, u_\varepsilon)$ converges weakly in $H_0^1(\Omega) \times L^2(\Omega)$ to some (y^0, u^0) which is feasible for $(\tilde{P}^\varepsilon)$. In addition the limit point satisfies*

$$\mathcal{J}(y^0, u^0) \leq \liminf_{\varepsilon\to 0} \mathcal{J}(y_\varepsilon, u_\varepsilon)$$

Proof. As already pointed out in the preceding result, $(0, -f)$ is an element of $\mathcal{D}_\varepsilon$ for any $\varepsilon > 0$ where $\mathcal{D}_\varepsilon$ represents the feasible set of (P^ε). Thus we obtain

$$\mathcal{J}(y_\varepsilon, u_\varepsilon) \leq \mathcal{J}(0, -f) = \frac{1}{2}|y_d|^2 + \frac{\beta}{2}|f|^2$$

which yields boundedness of $\{y_\varepsilon\}$ in $H_0^1(\Omega)$ and $\{u_\varepsilon\}$ in $L^2(\Omega)$. Consequently there exist elements $y^0 \in H_0^1(\Omega)$ and $u^0 \in L^2(\Omega)$ such that

$$y_\varepsilon \rightharpoonup y^0 \text{ in } H_0^1(\Omega), \;\; u_\varepsilon \rightharpoonup u^0 \text{ in } L^2(\Omega)$$

along subsequences again denoted by ε. Moreover, since $H_0^1(\Omega) \to L^2(\Omega)$ is compact for bounded domains, $y_\varepsilon \to y^0$ strongly in $L^2(\Omega)$. Now consider (VI^ε). Isolating the weighted Laplacian on the right hand side we obtain

$$\langle \mathbf{b} \cdot \nabla y_\varepsilon - b^0 y_\varepsilon - f - u_\varepsilon, v - y_\varepsilon \rangle \geq \varepsilon \langle \Delta y_\varepsilon, v \rangle - \varepsilon \langle \Delta y_\varepsilon, y_\varepsilon \rangle = \varepsilon \langle \Delta y_\varepsilon, v \rangle + \varepsilon \|y_\varepsilon\|^2 \geq -\varepsilon c \|v\|$$

for all $v \in \mathbf{K}$ with $c > 0$ constant. Due to the strong convergence in of y_ε in $L^2(\Omega)$, the limit elements satisfy

$$\langle \mathbf{b} \cdot \nabla y^0 - b^0 y^0 - f - u^0, v - y^0 \rangle \geq 0$$

for all $v \in \mathbf{K}$. Consequently, (y^0, u^0) solves (VI^0). The claimed estimate of $\mathcal{J}(y^0, u^0)$ stems from thee weak lower semicontinuity of $\mathcal{J}$. □

In a second step we prove, that the limit element (y^0, u^0) from the preceding result is optimal for $(\tilde{P}^\varepsilon)$ among all solutions in the viscosity sense as a Corollary of Theorem 6.1.2.

Corollary 6.1.1. *Any limit pairing $(y^0, u^0) \in H_0^1(\Omega) \times L^2(\Omega)$ from Proposition 6.1.4 is optimal for $(\tilde{P})$.*

Proof. Let (y, u) be an arbitrary pairing satisfying (VI^0) and being feasible for $(\tilde{P})$, i.e. $(y, u) \in H_0^1(\Omega) \times L^2(\Omega)$. According to Theorem 6.1.2, for any sequence of viscosity parameters $\varepsilon \to 0$, the pairings $(\tilde{y}_\varepsilon, u)$, with $\tilde{y}_\varepsilon$ solving (VI^ε) for u, contain a subsequence converging strongly to (y, u) in $H_0^1(\Omega) \times L^2(\Omega)$. Thus

$$\mathcal{J}(y^0, u^0) \leq \liminf_{\varepsilon \to 0} \mathcal{J}(y_\varepsilon, u_\varepsilon) \leq \lim_{\varepsilon \to 0} \mathcal{J}(\tilde{y}_\varepsilon, u) = \mathcal{J}(y, u)$$

where the second inequality sign comes from the optimality of $(y_\varepsilon, u_\varepsilon)$ for $(\tilde{P}^\varepsilon)$. Since there is no alternative in the feasible set of $(\tilde{P})$ realizing a better value of the objective, (y^0, u^0) has to be optimal for the problem. □

Considering the equivalent complementarity problem (P^ε) we have to take care of one further issue. Although Lemma 6.1.4 and 6.1.5 ensure the equivalence of the involved variational inequalities to complementarity problems, Proposition 6.1.4 only provides boundedness of the sequence $\{(y_\varepsilon, u_\varepsilon)\}$ in $H_0^1(\Omega) \times L^2(\Omega)$. Thus we can a priori only obtain the boundedness of

$$\xi_\varepsilon = -\varepsilon \Delta y_\varepsilon + \mathbf{b} \cdot \nabla y_\varepsilon + b^0 y_\varepsilon - f - u_\varepsilon$$

in $H^{-1}(\Omega)$. The next result allows for a better estimate of this quantity.

Proposition 6.1.5. *The sequence of slack variables $\{\xi_\varepsilon\}$ is bounded in $L^2(\Omega)$.*

Proof. By Lemma 6.1.2 and the assumptions on the obstacle, we can apply [133, Proposition 5:2:2] and obtain for any feasible pairing (y, u) and $\varepsilon > 0$ the estimate

$$|A^\varepsilon y_\varepsilon| \leq |f + u_\varepsilon| + |(-f - u_\varepsilon)^+|$$

providing an upper bound on the image $A^\varepsilon y_\varepsilon$ in $L^2(\Omega)$. In addition, this upper bound is independent of ε. Consequently, any term in the equation defining ξ_ε is an element of $L^2(\Omega)$ and so is ξ_ε. Moreover, the right hand side is bounded in $L^2(\Omega)$ for a sequence $(y_\varepsilon, u_\varepsilon)$ of minimizers for the problems $(\tilde{P}^\varepsilon)$ and so is the sequence $\{\xi_\varepsilon\}$. $\square$

By the established bound, ξ_ε converges weakly in $L^2(\Omega)$ along a subsequence. Note that this result is the reason for the high regularity we assumed the obstacle to have.

The Boundedness of y_ε in $H_0^1(\Omega)$

We have seen in the preceding section, that a bound on the weak gradients of the solution to the approximating variational inequalities is crucial for the convergence to solutions of (VI^0). A further possibility for obtaining such bounds is the consideration of another convex set defining the variational inequality. Instead of the obstacle problem, let the set $\mathbf{K}$ be given as

$$\tilde{\mathbf{K}} = \{\phi \in H_0^1(\Omega) | |\nabla\phi| \leq c \text{ a.e.}\}$$

for some strictly positive constant $c > 0$. First we note, that $\tilde{\mathbf{K}}$ is convex and closed and thus weakly closed. Standard theory [58, 94] provides the existence of unique solutions $y_\varepsilon \in H_0^1(\Omega)$ to (VI^ε) when $\mathbf{K}$ is replaced by $\tilde{\mathbf{K}}$. Considering a sequence of viscosity parameters $\varepsilon \to 0$ and fixing some data $f \in L^2(\Omega)$ the sequence $\{y_\varepsilon\}$ is bounded in $H_0^1(\Omega)$ due to the nature of the constraint set and contains a weakly converging subsequence with limit element $y \in H_0^1(\Omega) \cap \tilde{\mathbf{K}}$ by the weak closedness of $\tilde{\mathbf{K}}$ and y solving (VI^0) for f. Regarding existence of solutions of quasivariational inequalities this behavior has been investigated in several papers ([80, 81, 136]). However the difficult nature of first order differential operators defining the constraint set of the variational inequalities should not be underestimated. In particular the last reference includes an elementary gap in the process deriving solutions to time dependent first order variational inequalities which to our knowledge has been fixed only very recently.

While proving the existence of solutions with a certain kind of regularity to the first order variational inequalities becomes straight forward in in the presence of $\tilde{\mathbf{K}}$, the optimal control of such objects becomes much more complicated even in the PDE constraint case (e.g. see [33, 78, 132]) and especially in the case of variational inequalities as studied in [82]. If the upper bound on the gradient is a constant, one might apply a transformation technique as in [58] to obtain an obstacle problem instead of the gradient constraint one but this would be a very restrictive assumption.

Instead of the Tikhonov like term we decided for, any general convex $\mathcal{J}$ containing a quantification of y in $H_0^1(\Omega)$ can be used. However, since y^d is usually represents measured data, it is not to be expected that a term like $\|y - y^d\|$ can be evaluated. Thus a further possibility would be the $H_0^1(\Omega)$ misfit of the state y and a mollification of y^d.

For completeness we mention, that the bound on $\|y_\varepsilon\|$ to (VI^ε) can also be achieved by an explicit direct state constraint in $(\tilde{P}^{\varepsilon})$, $(\tilde{P})$, namely

$$\|y_{\varepsilon}\|^2 \leq c$$

Unfortunately, this kind of constraint did not yield satisfying results in the context of stationarity systems. An estimate for the corresponding Lagrange multiplier was not possible in the convergence process for the stationarity systems of the penalized regularized problems introduced below to the stationarity system of $(\tilde{P}^{\varepsilon})$.

6.2 Stationarity Results for the Control Problem

In this section we establish stationarity results for (P). This is done directly on the hyperbolic level and by the already mentioned vanishing viscosity approach. In the latter case, for any $\varepsilon > 0$ we can utilize existing theory established in [73, 95]. Thus we will skip most of the corresponding proof restricting ourselves to remarks on the changes for our setting. For the results on the hyperbolic level we present the proofs to show, that the nature of the differential operator does not effect the outcome. The line of arguments follows mostly [73].
As introduced in Section 3.3, we will use a penalization-regularization approach for the derivation of stationarity conditions.

6.2.1 Results on the Hyperbolic Level

Recall problem (P). We will apply a penalization technique for the nonegativity condition of the state parametrized by γ. In addition we regularize the equality constraint on the inner product of y and ξ by *inflating* the feasible set. This approach has already been discussed in Section 3.3.3 and for elliptic problems the theory was established in [73]. We obtain the following problem.

$$\begin{array}{lrl} \min & \mathcal{J}(y,u) + \frac{1}{2\gamma}|(\bar{\lambda} - \gamma y)^+|^2 & = \mathcal{J}_\gamma(y,u) \\ \text{s.t.} & \mathbf{b}\cdot\nabla y + b^0 y - f - u & = \xi \\ & \xi \geq 0, (\xi, y) & \leq \alpha \end{array} \tag{$P^0_{\gamma,\alpha}$}$$

Here, $0 \leq \bar{\lambda} \in L^p(\Omega)$ with $p > 2$ represents a nonnegative threshold. In practice this quantity is mostly chosen to be zero.
First we establishes solvability of this approximating problem.

Proposition 6.2.1. *For arbitrary $(\gamma, \alpha) > 0$, $(P^0_{\gamma,\alpha})$ admits a solution.*

Proof. Let

$$\mathcal{D}_{\gamma,\alpha} = \{(y,u,\xi) \in H^1_0(\Omega) \times L^2(\Omega) \times L^2(\Omega) | A^0 y - f - u - \xi = 0, \xi \geq 0, (y,\xi) \leq \alpha\}$$

denote the feasible set. We have already constructed a feasible point $(\tilde{P})$ which is feasible for any (γ, α) as well. Thus, $\mathcal{D}_{\gamma,\alpha} \neq \emptyset$. The norm character of $\mathcal{J}_\gamma$ ensures boundedness from below and coercivity of the objective functional. Consider an infimizing sequence $(y_n, u_n, \xi_n) \in \mathcal{D}_{\gamma,\alpha}$ with

$$\lim_{n\to\infty} \mathcal{J}_\gamma(y_n, u_n, \xi_n) = \inf_{(y,u,\xi)\in\mathcal{D}} \mathcal{J}_\gamma(y,u,\xi) = a.$$

By coercivity of $\mathcal{J}_\gamma$ we find $|u_n|$ and $\|y_n\|$ to be bounded by a constant c only depending on y^d, f, γ, β and $\tilde{\beta}$. Moreover, the state equation, triangle inequality and Hölder inequality provide

$$\begin{aligned}|\xi_n| = |A^0 y_n - f - u_n| &\leq \sum_{i=1}^n |b^i|_\infty |y_{n_{x_i}}| + |b^0|_\infty |y_n| + |f| + |u_n| \\ &\leq (n+1) \max_{0 \leq i \leq n} \{|b^i|_\infty\} \|y_n\| + |u_n| + |f|\end{aligned}$$

So ξ_n is bounded in $L^2(\Omega)$ by a constant c in addition depending on b and there exist a subsequence such that

$$y_n \rightharpoonup \tilde{y} \text{ in } H_0^1(\Omega), \quad u_n \rightharpoonup \tilde{u} \text{ in } L^2(\Omega) \quad \text{and} \quad \xi_n \rightharpoonup \tilde{\xi} \text{ in } L^2(\Omega)$$

with limit elements $(\tilde{y}, \tilde{u}, \tilde{\xi}) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$. Using this convergence properties in the state equation we find for all $\phi \in L^2(\Omega)$

$$\lim_{n \to \infty} (A^0 y_n - f - u_n - \xi_n, \phi) = (A^0 \tilde{y} - f - \tilde{u} - \tilde{\xi}, \phi) = 0.$$

Since the embedding $H_0^1(\Omega) \to L^2(\Omega)$ is compact, we find for a further subsequence denoted by (y_n, ξ_n)

$$\lim_{n \to \infty} (y_n, \xi_n) = (\tilde{y}, \tilde{\xi}) \leq \alpha$$

ensuring, that the condition on the $L^2(\Omega)$ product of $\tilde{y}$ and $\tilde{\xi}$ is satisfied. Finally,

$$K = \{v \in L^2(\Omega) | v(x) \geq 0 \text{ for almost every } x \in \Omega\}$$

is weakly closed (see Theorem A.4.1). Consequently, $\tilde{\xi} \geq 0$ almost everywhere is satisfied and $(\tilde{y}, \tilde{u}, \tilde{\xi}) \in \mathcal{D}$ is proven. Since the objective is weakly lower semicontinuous (see Theorem A.4.4) we find

$$a = \liminf_{n \to \infty} \mathcal{J}_\gamma(y_n, u_n, \xi_n) \geq \mathcal{J}_\gamma(\tilde{y}, \tilde{u}, \tilde{\xi}) \geq a$$

ensuring that the infimum is attained. □

After establishing the existence of solutions for the auxiliary problems, we have to prove, that a sequence of solutions converges to a solution of the original problem.

Proposition 6.2.2. *Consider a sequence $\gamma \to \infty$ with $\gamma \geq \underline{\gamma} > 0$ and a coupled sequence $\alpha = \alpha(\gamma) \to 0$. Let $(y_\gamma, u_\gamma, \xi_\gamma)$ denote corresponding solutions to the problems $(P^0_{\gamma,\alpha})$. Then there exist a subsequence $\gamma \to \infty$ again denoted by γ and*

$$(y^*, u^*, \xi^*) \in L_B(\Omega) \cap H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$$

such that

$$y_\gamma \rightharpoonup y^* \text{ in } H_0^1(\Omega), u_\gamma \rightharpoonup u^* \text{ in } L^2(\Omega) \text{ and } \xi_\gamma \rightharpoonup \xi^* \text{ in } L^2(\Omega).$$

In addition (y^*, u^*, ξ^*) is a solution of (P).

Proof. $(y,u,\xi)=(0,-f,0)$ is feasible for any combination of (γ,α). Consequently we find

$$\mathcal{J}(y_\gamma,u_\gamma)\le\mathcal{J}_\gamma(y_\gamma,u_\gamma)\le\mathcal{J}_\gamma(0,-f,0)\le\frac{1}{2}|y^d|^2+\frac{\beta}{2}|f|^2+\frac{1}{2\underline{\gamma}}|\bar\lambda|^2 \tag{6.6}$$

providing an upper bound c for $\|y_\gamma\|$ and $|u_\gamma|$ independent of γ and α. As in Proposition 6.2.1 we find ξ_γ to be bounded independent of γ and α as well. Thus there exists a subsequence with

$$y_\gamma\rightharpoonup y^* \text{ in } H_0^1(\Omega),\quad u_\gamma\rightharpoonup u^* \text{ in } L^2(\Omega)\quad\text{and}\quad \xi_\gamma\rightharpoonup\xi^* \text{ in } L^2(\Omega)$$

where $y_\gamma\to y^*$ strongly in $L^2(\Omega)$ due to the compact embedding of $H_0^1(\Omega)\to L^2(\Omega)$. By the obtained convergence properties we find for every $\phi\in L^2(\Omega)$

$$\lim_{\gamma\to\infty}(A^0y_\gamma-f-u_\gamma-\xi_\gamma,\phi)=(A^0y^*-f-u^*-\xi^*,\phi)=0.$$

Since $\mathbf{K}$ as defined in the preceding proof is weakly closed,

$$\xi^*\ge 0 \text{ a.e.}$$

holds. From the strong convergence of y_γ in $L^2(\Omega)$ we find

$$(y^*,\xi^*)=\lim_{\gamma\to\infty}(y_\gamma,\xi_\gamma)\le\lim_{\gamma\to\infty}\alpha(\gamma)=0.$$

Considering (6.6) we in addition obtain the estimate

$$0\le|(\bar\lambda/\gamma-y_\gamma)^+|^2\le c/\gamma$$

and since $c/\gamma\to 0$ in the limit process $(\bar\lambda/\gamma-y_\gamma)^+\to 0$ in $L^2(\Omega)$ follows directly. By the Lemma of Fatou (Theorem A.4.11) we further obtain

$$\int_\Omega((-y^*)^+)^2\le\int_\Omega\liminf_{\gamma\to\infty}(\bar\lambda/\gamma-y_\gamma)^+)^2\le\liminf_{\gamma\to\infty}\int_\Omega((\bar\lambda/\gamma-y_\gamma)^+)^2\le\lim\left|\left(\frac{\bar\lambda}{\gamma}-y_\gamma\right)^+\right|^2_{L^2(\Omega)}=0$$

implying $y^*\ge 0$ almost everywhere. Consequently, y^* and ξ^* satisfy the complementarity condition and we have proven, that (y^*,u^*,ξ^*) is feasible for (P). In order to prove optimality of the limit point we consider $(\tilde y,\tilde u,\tilde\xi)\in H_0^1(\Omega)\times L^2(\Omega)\times L^2(\Omega)$ optimal for (P). Then the estimates $\mathcal{J}(\tilde y,\tilde u)\le\mathcal{J}(y^*,u^*)$ and $\mathcal{J}_\gamma(y_\gamma,u_\gamma)\le\mathcal{J}_\gamma(\tilde y,\tilde u)$ hold by optimality. This provides

$$\mathcal{J}(\tilde y,\tilde u)\le\mathcal{J}(y^*,u^*)\le\liminf_{\gamma\to\infty}\mathcal{J}(y_\gamma,u_\gamma)\le\liminf_{\gamma\to\infty}\mathcal{J}_\gamma(y_\gamma,u_\gamma)\le\limsup_{\gamma\to\infty}\mathcal{J}_\gamma(\tilde y,\tilde u)=\mathcal{J}(\tilde y,\tilde u)$$

implying $\mathcal{J}(y^*,u^*)=\mathcal{J}(\tilde y,\tilde u)$ therefore, (y^*,u^*) is optimal. $\square$

The next result establishes a well defined first order optimality system for $(P^0_{\gamma,\alpha})$.

Proposition 6.2.3. *Let* $(y,u,\xi) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ *be an optimal solution of* $(P^0_{\gamma,\alpha})$ *for some* $(\gamma,\alpha) > 0$.
Then there exist Lagrange multipliers $(p,\mu,r) \in L^2(\Omega) \times L^2(\Omega) \times \mathbb{R}$ *satisfying the system*

$$y - y^d - \tilde{\beta}\Delta y - (\bar{\lambda} - \gamma y)^+ + (A^0)^* p + r\xi = 0 \tag{6.7a}$$
$$\beta u - p = 0 \tag{6.7b}$$
$$ry - p - \mu = 0 \tag{6.7c}$$
$$\xi \geq 0, \quad \mu \geq 0, \quad (\xi,\mu) = 0 \tag{6.7d}$$
$$\alpha \geq (\xi, y), \quad r \geq 0, \quad r(\alpha - (\xi,y)) = 0 \tag{6.7e}$$
$$A^0 y - f - u - \xi = 0 \tag{6.7f}$$

Proof. The proof is a direct application of Theorem 3.1.2 with

$$\begin{aligned} X = C &= H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \\ f(x) &= \mathcal{J}_\gamma(y,u) \\ Y &= L^2(\Omega) \times L^2(\Omega) \times \mathbb{R} \\ K &= \{0\} \times L^2(\Omega)^+ \times \mathbb{R}^+ \\ g(x) &= (-A^0 y + f + u + \xi, \xi, \alpha - (y,\xi)) \end{aligned}$$

Consider an optimal solution $(\bar{y},\bar{u},\bar{\xi})$ of (P). For the application of the general result from Zowe and Krucyusz we have to find for any combination $(y_1,y_2,y_3) \in Y$ some elements $(c_1,c_2,c_3) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$, a factor $\lambda > 0$ and $(k_2,k_3) \in L^2(\Omega)^+ \times \mathbb{R}^+$ such that

$$\underbrace{\begin{pmatrix} -A^0 & 1 & 1 \\ 0 & 0 & 1 \\ -(\cdot,\bar{\xi}) & 0 & -(\bar{y},\cdot) \end{pmatrix}}_{=g'(\bar{y},\bar{u},\bar{\xi})} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} - \begin{pmatrix} 0 \\ k_2 - \lambda\bar{\xi} \\ k_3 - \lambda(\alpha - (\bar{y},\bar{\xi})) \end{pmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$$

is satisfied. Here we have to consider two cases.
In case of $\bar{\xi} = 0$ the choice $c_1 = 0, c_2 = y_1 - y_2, c_3 = y_2, k_2 = 0, k_3 = -(y_3 + (\bar{y},y_2))^-, \lambda = (y_3 + (\bar{y},y_2))^+/\alpha$ meets the requirements.
In case of $\bar{\xi} \neq 0$ we first set $\lambda = 0, k_2 = 0, c_3 = y_2, k_3 = 0$ and then the remaining variables c_1 and c_2 have to satisfy

$$-A^0 c_1 + c_2 = y_1 - y_2, \quad -(\bar{\xi}, c_1) = y_3 + (\bar{y}, y_2).$$

Since $\bar{\xi} \neq 0$, there exists a function $w \in H_0^1$ with $-(\bar{\xi},w) = y_3 + (\bar{y},y_2)$. Thus, $c_1 = w$, $c_2 = y_1 - y_2 + A^0 w$ is an element of $L^2(\Omega)$ and the requirements are met as well. $\square$

Now we will demonstrate what kind of stationarity system can be derived from this optimality conditions. Therefore we introduce the following parameter depending subsets of Ω.

- $N_\gamma = \{x \in \Omega | y_\gamma < 0\}$
- $\Lambda_\gamma = \{x \in \Omega | \bar{\lambda}(x) - \gamma y_\gamma > 0\}$

Moreover we recall a technical Lemma from [73] which is independent of the differential operator.

Lemma 6.2.1. *Let* $\{y_\gamma\} \in L^2(\Omega)$ *be a sequence such that* $\{|(y_\gamma, (\bar{\lambda} - \gamma y_\gamma)^+)|\}$ *is bounded. Then we have*

a) There exists $c > 0$ *independent of* γ *such that* $\gamma \int_{N_\gamma} y_\gamma^2 \leq c$ *for all* $\gamma > 0$

b) $\limsup_{\gamma \to \infty} (y_\gamma, (\bar{\lambda} - \gamma y_\gamma)^+) \leq 0$

Finally, we present the result for the limit process of the first order optimality systems (6.7).

Theorem 6.2.1. *Consider a sequence of penalization parameters with* $\gamma \to 0$ *and a coupled sequence* $\alpha = \alpha(\gamma) \to 0$. *Let* $(y_\gamma, u_\gamma, \xi_\gamma) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$ *be a sequence of solutions to* (6.7) *for the corresponding parameters which is bounded in* $H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$.
Then there exist a subsequence of parameters again denoted by γ *and elements*

$$(y^*, u^*, \xi^*, p^*, \lambda^*) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega)$$

such that

$$\begin{gathered} y_\gamma \rightharpoonup y^* \text{ in } H_0^1(\Omega), u_\gamma \rightharpoonup u^* \text{ in } L^2(\Omega), \xi_\gamma \rightharpoonup \xi^* \text{ in } L^2(\Omega) \\ p_\gamma \rightharpoonup p^* \text{ in } L^2(\Omega), (\bar{\lambda} - \gamma y_\gamma)^+ - r_\gamma \xi_\gamma \rightharpoonup \lambda^* \text{ in } H^{-1}(\Omega) \end{gathered}$$

satisfying the following system.

$$y^* - y^d - \tilde{\beta} \Delta y^* + A^0 p^* + \lambda^* = 0 \tag{6.8a}$$
$$\beta y^* - p^* = 0 \tag{6.8b}$$
$$A^0 y^* - u^* - \xi^* = 0 \tag{6.8c}$$
$$y^* \geq 0,\ \xi^* \geq 0,\ (y^*, \xi^*) = 0 \tag{6.8d}$$
$$\forall \tau > 0 \exists E_\tau \subset \Omega_+ : |\Omega_+ \backslash E_\tau| \leq \tau, \forall \varphi \in H_0^1(\Omega), \varphi = 0 \text{ a.e. on } \Omega \backslash E_\tau : \langle \lambda^*, \varphi \rangle = 0 \tag{6.8e}$$

Proof. The assumed boundedness of the sequence $(y_\gamma, u_\gamma, \xi_\gamma)$ provides the existence of a subsequence again indexed by γ such that

$$y_\gamma \rightharpoonup y^* \text{ in } H_0^1(\Omega),\ u_\gamma \rightharpoonup \text{ in } L^2(\Omega),\ \xi_\gamma \rightharpoonup \text{ in } L^2(\Omega)$$

for limit elements $(y^*, u^*, \xi^*) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$. p_γ is bounded in $L^2(\Omega)$ due to (6.7b). Consequently, there exist a weakly converging subsequence converging to some $p^* \in L^2(\Omega)$ and the corresponding limit element satisfies (6.8b). Similar to the proof of Proposition 6.2.2 we find (6.8c), $\xi^* \geq 0$ and $(y^*, \xi^*) \leq 0$ to be satisfied by the limit elements. In order to establish a bound for $r_\gamma(\xi_\gamma, y_\gamma)$ we test (6.7c) by ξ_γ and find

$$0 \leq r_\gamma \alpha_\gamma = r_\gamma(\xi_\gamma, y_\gamma) = (p_\gamma, \xi_\gamma) + (\mu_\gamma, \xi_\gamma) \leq |p_\gamma||\xi_\gamma|.$$

Testing (6.7a) with y_γ we thus obtain

$$|((\bar{\lambda} - \gamma y_\gamma)^+, y_\gamma)| \leq |A^0 y_\gamma||p_\gamma| + |y_\gamma|^2 + \tilde{\beta}\|y_\gamma\|^2 + |y_\gamma||y^d| + |\xi_\gamma||y_\gamma| + |p_\gamma||\xi_\gamma|$$

and so the left hand side is bounded. Moreover, the estimate

$$|(\gamma)^{-1/2}(\bar{\lambda} - \gamma y_\gamma)^+|^2 \leq \frac{1}{\gamma} \int\limits_\Omega \bar{\lambda}^2 - 2 \int\limits_{N_\gamma} \bar{\lambda} y_\gamma + \gamma \int\limits_{N_\gamma} y_\gamma^2$$

holds true providing boundedness of $(\gamma)^{-1/2}(\bar{\lambda} - \gamma y_\gamma)^+$ in $L^2(\Omega)$ under consideration of Lemma 6.2.1 a). Similar to Proposition 6.2.2 we now obtain

$$y^* \geq 0 \text{ a.e.}$$

proving, that the limit elements satisfy the complementarity condition (6.8c) - (6.8d). Next we establish the weak convergence of $(\bar{\lambda} - \gamma y_\gamma)^+ - r_\gamma \xi_\gamma$. From (6.7a) we find for $\varphi \in H_0^1(\Omega)$

$$\begin{aligned} \sup_{\|\varphi\|=1} |((\bar{\lambda} - \gamma y_\gamma)^+ - r_\gamma \xi_\gamma, \varphi)| &= \sup_{\|\varphi\|=1} |(A^0\varphi, p_\gamma) + (y_\gamma - y^d, \varphi) + \tilde{\beta}(\nabla y_\gamma, \nabla \varphi)| \\ &\leq \sup_{\|\varphi\|=1} c|p_\gamma| + \sup_{\|\varphi\|=1} |y_\gamma - y^d||\varphi| + \sup_{\|\varphi\|=1} \tilde{\beta}\|y_\gamma\|\|\varphi\| \leq c \end{aligned}$$

and therefore boundedness of $\{(\bar{\lambda} - \gamma y_\gamma)^+ - r_\gamma \xi_\gamma\}$ in $H^{-1}(\Omega)$. Thus there exist a limit element λ^* and a weakly converging subsequence again denoted by γ satisfying

$$(\bar{\lambda} - \gamma y_\gamma)^+ - r_\gamma \xi_\gamma \rightharpoonup \lambda^* \text{ in } H^{-1}(\Omega).$$

Moreover, (6.8a) holds in $H^{-1}(\Omega)$ for y^*, p^* and λ^*. To prove (6.8e) we observe that due to the compact embedding $H_0^1(\Omega) \to L^2(\Omega)$ y_γ converges strongly in $L^2(\Omega)$ and due to Theorem A.4.9 pointwise almost everywhere in Ω along a subsequence denoted the same. Thus for almost every $x \in \Omega_+$ we find $\bar{\lambda}(x) - \gamma y_\gamma(x) < 0$ for γ sufficiently large. Consequently we obtain

$$\bar{\lambda} - \gamma y_\gamma \to 0 \text{ pointwise in } \Omega_+.$$

Theorem A.4.10 now ensures for any given $\tau > 0$ the existence of some set $E_\tau \subset \Omega_+$ with $|\Omega_+ \backslash E_\tau| \leq \tau$ and

$$(\bar{\lambda} - \gamma y_\gamma) \to 0 \text{ uniformly on } E_\tau.$$

Finally, for every $\tau > 0$ this provides

$$\langle \lambda^*, \varphi \rangle = \lim_{\gamma \to \infty} ((\bar{\lambda} - \gamma y_\gamma)^+, \varphi) = 0$$

for all $\varphi \in H_0^1(\Omega)$ satisfying $\varphi(x) = 0$ on $\Omega \backslash E_\tau$ showing (6.8e). □

6.2.2 Results obtained by the vanishing Viscosity Approach

Stationarity of the Auxiliary Problem

After ensuring the existence of minimizers for the optimization problem subject to an elliptic variational inequality we now focus on the characterization of stationary points for those problems. Following the procedure carried out in Section 3.3 and taking into account the slack quantity ξ we end up with

$$\begin{array}{llll} \min & |y-y^d|^2 + \frac{\tilde{\beta}}{2}\|y\|^2 + \frac{\beta}{2}|u|^2 + \frac{1}{2\gamma}|(\bar{\lambda}-\gamma y)^+|^2 + \frac{\kappa}{2}|\xi|^2 & = \mathcal{J}_\gamma(y,u,\xi) & \\ \text{s.t.} & -\varepsilon\Delta y + \mathbf{b}\cdot\nabla y + b^0 y - f - u & = \xi & (\tilde{P}^\varepsilon_{\gamma,\alpha,\kappa}) \\ & \xi \geq 0, (\xi,y) & \leq \alpha & \end{array}$$

As mentioned, we can utilize existing theory. Therefore we omit the proofs of the following results since they are almost identical to the theory developed in [73, 95].
The existence of minimizers for the relaxed regularized problems $(\tilde{P}^\varepsilon_{\gamma,\alpha,\kappa})$ is established by the following result.

Proposition 6.2.4. *For every $(\gamma,\alpha,\gamma) > 0$ problem $(\tilde{P}^\varepsilon_{\gamma,\alpha,\kappa})$ admits a solution.*

Next we discuss a consistency of the relaxed-regularized problems with the auxiliary problems.

Theorem 6.2.2. *For fixed $\varepsilon > 0$ and $(\alpha,\gamma,\kappa) \to (0,\infty,0)$ we consider a sequence of solutions to the problems $(\tilde{P}^\varepsilon_{\gamma,\alpha,\kappa})$.*
Then there exist an optimal triplet $(y^*,u^*,\xi^*) \in H^1_0(\Omega)\times L^2(\Omega)\times L^2(\Omega)$ for (P^ε) such that

$$y_{(\alpha,\gamma,\kappa)} \to y^* \text{ in } H^1_0(\Omega), u_{(\alpha,\gamma,\kappa)} \to u^* \text{ in } L^2(\Omega), \xi_{(\alpha,\gamma,\kappa)} \to \xi^* \text{ in } H^{-1}(\Omega).$$

In addition the regualrization terms $(1/2\gamma)\|(\bar{\lambda}-\gamma y_{(\alpha,\gamma,\kappa)})\|^2$ and $(\kappa/2)|\xi_{(\alpha,\gamma,\kappa)}|$ tend to zero as $\gamma\to\infty$.

We point out, that for certain domains the state y^* gains regularity and is an element of $H^1_0(\Omega)\cap H^2(\Omega)$ (see Theorem 2.3.1 and the comments below). Moreover we can characterize stationary points of the problems $(\tilde{P}^\varepsilon_{\gamma,\alpha,\kappa})$ according to the following result.

Proposition 6.2.5. *Let $(y,u,\xi) \in H^1_0(\Omega)\times L^2(\Omega)\times L^2(\Omega)$ be an optimal solution for $(\tilde{P}^\varepsilon_{\gamma,\alpha,\kappa})$ for some $(\alpha,\kappa,\gamma) > 0$ and fixed $\varepsilon > 0$.*
Then there exist Lagrange multipliers $(p,\mu,r) \in H^1_0(\Omega)\times L^2(\Omega)\times\mathbb{R}$ satisfying

$$y - y^d - \tilde{\beta}\Delta y - (\bar{\lambda}-\gamma y)^+ + (A^\varepsilon)^* p + r\xi = 0 \tag{6.9a}$$
$$\beta u - p = 0 \tag{6.9b}$$
$$\kappa\xi - p - \mu + ry = 0 \tag{6.9c}$$
$$\xi \geq 0 \quad \mu \geq 0 \quad (\xi,\mu) = 0 \tag{6.9d}$$
$$\alpha \geq (\xi,y) \quad r \geq 0 \quad r(\alpha - (\xi,y)) = 0 \tag{6.9e}$$
$$A^\varepsilon y - f - u - \xi = 0 \tag{6.9f}$$

The preceding result yields almost the same stationarity system as in [95] since the constraint set is unchanged in the problems considered in this thesis and only the Fréchet derivative of the objective differs providing a slightly less regular adjoint equation (6.9a).
After establishing the first order necessary optimality condition, the stationary system for (P^ε) is derived on this basis. The proof (see [73]) only works for a certain update strategy of the parameters α and κ depending on γ. Thus, to keep the notation clear, we will only use γ in the further discussion. The update rule is a given as follows.

Assumption 6.1. *Let $\gamma \to \infty$ be given. The sequence $(\alpha_\gamma, \kappa_\gamma) \to 0$ satisfies the update strategy if*

$$\max\{(\alpha_\gamma\sqrt{\gamma})^{-1}, \kappa_\gamma\sqrt{\gamma}\} \le C$$

holds independently of γ.

Again we point out, that the proof of the following result is almost identical to already established theory as in [73, 95]. The only change comes from the less regular right hand side in the adjoint equation (6.9a). The minor modification of the proof not justify a complete repetition. The stationarity system for (P^ε) is defined in the following result

Theorem 6.2.3. *Assume, that (u_γ, ξ_γ) is uniformly bounded in $L^2(\Omega) \times L^2(\Omega)$. Then there exist*

$$(y^*, u^*, \xi^* \lambda^*, p^*) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega) \times H_0^1(\Omega)$$

such that

$$\begin{aligned} y_\gamma \to y^* \text{ in } H_0^1(\Omega),\ \ u_\gamma \rightharpoonup u^* \text{ in } H_0^1(\Omega),\ \ \xi_\gamma \rightharpoonup \xi^* \text{ in } L^2(\Omega), \\ (\bar\lambda - \gamma y_\gamma)^+ - r_\gamma \xi_\gamma \rightharpoonup \lambda^* \text{ in } H^{-1}(\Omega),\ \ p_\gamma \rightharpoonup p^* \text{ in } H_0^1(\Omega) \end{aligned}$$

satisfying the system

$$(A^\varepsilon)^* p^* + y^* - y^d - \tilde\beta \Delta y^* - \lambda^* = 0 \tag{6.10a}$$
$$\beta u^* - p^* = 0 \tag{6.10b}$$
$$A^\varepsilon y^* - f - u^* - \xi^* = 0 \tag{6.10c}$$
$$\xi^* \ge 0, \quad y^* \ge 0, \quad (\xi^*, y^*) = 0 \tag{6.10d}$$
$$p^*(x) = 0 \text{ for almost every } x \text{ s.t. } \xi^*(x) > 0 \tag{6.10e}$$
$$\langle \lambda^*, y^* \rangle = 0 \tag{6.10f}$$
$$\langle \lambda^*, p^* \rangle \le 0 \tag{6.10g}$$

Moreover, for every $\tau > 0$ there exists a set $E_\tau \subset \Omega_+$ with $|\Omega_+ \backslash E_\tau| \le \tau$ such that

$$\langle \lambda^*, v \rangle = 0 \text{ for all } v \in H_0^1(\Omega),\ \ v = 0 \text{ a.e. in } \Omega \backslash E_\tau, \tag{6.11}$$

We recall, that under certain conditions additional convergence assumptions, the limit points are stationary points of higher regularity. The proof can be found in [73]. We introduce the biactive set

$$B = \{x \in \Omega : y^*(x) = 0\} \cap \{x \in \Omega : \xi^*(x) = 0\}.$$

Lemma 6.2.2. *Let the assumptions of Theorem 6.2.3 hold true.*

a) *If $\langle \lambda_\gamma, y^* \rangle \to 0$ or equivalently $r_\gamma(\xi_\gamma, y^*) \to 0$ for $\gamma \to \infty$, then (y^*, u^*, ξ^*) is almost C-stationary for $(\tilde P^\varepsilon_{\gamma,\alpha,\kappa})$.*

b) *If Ω_+ is a Lipschitz domain, then (y^*, u^*, ξ^*) is C-stationary*

c) *If $r_\gamma(y_\gamma, w)_{L^2(B)} \to 0$ for all $w \in L^2(\Omega)$ and $r_\gamma(\xi_\gamma, v)_{L^2(B\cup\Omega_+)} \to 0$ for all $v \in H_0^1(\Omega)$ then (y^*, u^*, ξ^*) is $\mathcal{E}$-almost S-stationary. Furthermore, if (y^*, u^*, ξ^*) is almost C-stationary or C-stationary, then the assumptions imply almost S-stationarity and S-stationarity respectively.*

Limit of the Stationarity System

In this section we discuss the limit of solutions to the stationarity systems (6.10) for $\varepsilon \to 0$ under certain conditions on the boundedness of the sequence of solutions.

Theorem 6.2.4. *Consider a sequence of viscosity parameters ε converging to zero and corresponding solutions to the stationarity systems* (6.10)*.*

If the sequence $(y_\varepsilon, u_\varepsilon, \xi_\varepsilon)$ is bounded in $H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega)$, then there exist weak limit points

$$(y^*, u^*, \xi^*, p^*, \lambda^*) \in H_0^1(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times L^2(\Omega) \times H^{-1}(\Omega)$$

with

$$y_\varepsilon \rightharpoonup y^* \text{ in } H_0^1(\Omega), \;\; u_\varepsilon \rightharpoonup u^* \text{ in } L^2(\Omega), \quad \xi_\varepsilon \rightharpoonup \xi^* \text{ in } L^2(\Omega)$$
$$p_\varepsilon \rightharpoonup p^* \text{ in } L^2(\Omega), \;\; \lambda_\varepsilon + \varepsilon\Delta p_\varepsilon \rightharpoonup \lambda^* \text{ in } H^{-1}(\Omega)$$

satisfying the following system.

$$y^* - y_d - \tilde{\beta}\Delta y^* + (A^0)^* p^* - \lambda^* = 0 \tag{6.12a}$$
$$\beta u^* - p^* = 0 \tag{6.12b}$$
$$A^0 y^* - u^* - \xi^* - f = 0 \tag{6.12c}$$
$$y^* \geq 0 \quad \xi^* \geq 0 \quad (y^*, \xi^*) = 0 \tag{6.12d}$$
$$\text{on } S = \{x \in \Omega : \xi^*(x) > 0\} \text{ we have } \lim_{\varepsilon \to 0} (p_\varepsilon, \xi_\varepsilon) = 0 \tag{6.12e}$$
$$\text{for all } \tau > 0,\ \exists\, \Omega^\tau : |\Omega_+ \backslash \Omega^\tau| \leq \tau \text{ s.t. } \forall v \in C_c^\infty(\Omega), \;\; v = 0 \text{ a.e. in } \Omega \backslash \Omega^\tau : \; \langle \lambda^*, v \rangle = 0 \tag{6.12f}$$

Proof. The assumed boundedness of the sequence of stationary points provides weakly converging subsequences again denoted by ε and corresponding limit elements with

$$y_\varepsilon \rightharpoonup y^* \text{ in } H_0^1(\Omega), \;\; u_\varepsilon \rightharpoonup u^* \text{ in } L^2(\Omega), \;\; \xi_\varepsilon \rightharpoonup \xi^* \text{ in } L^2(\Omega).$$

The coupling (6.10b) provides boundedness of p_ε in $L^2(\Omega)$ and consequently the existence of a further subsequence and a limit element p^* with

$$p_\varepsilon \rightharpoonup p^* \text{ in } L^2(\Omega).$$

Now (6.12b) has to be satisfied by the limit elements u^* and p^*. Similar to the consistency proof in Proposition 6.2.2 we obtain for $\varepsilon \to 0$

$$A^0 y^* - u^* - \xi^* - f = 0 \text{ and } \xi^* \geq 0$$

and the same arguments providing nonnegativity of ξ^* yield

$$y^* \geq 0.$$

Moreover, the compact embedding $H_0^1(\Omega) \to L^2(\Omega)$ implies the existence of a further subsequence with

$$\lim_{\varepsilon \to 0} (\xi_\varepsilon, y_\varepsilon) = (\xi^*, y^*) = 0.$$

Thus (6.12c) and (6.12d) hold. Considering (6.10a) we find

$$\begin{aligned}
\sup_{\|\varphi\|=1} \langle \lambda_\varepsilon + \varepsilon \Delta p_\varepsilon, \varphi \rangle &= \sup_{\|\varphi\|=1} \langle -\mathbf{b} \cdot \nabla p_\varepsilon + \tilde{b}^0 p_\varepsilon + y_\varepsilon - y^d - \tilde{\beta} \Delta y_\varepsilon, \varphi \rangle \\
&\leq \sup_{\|\varphi\|=1} (\mathbf{b} \cdot \nabla \varphi + b^0 \varphi, p_\varepsilon) + \sup_{\|\varphi\|=1} (y_\varepsilon - y^d, \varphi) + \sup_{\|\varphi\|=1} \tilde{\beta} (\nabla y_\varepsilon, \nabla \varphi) \\
&\leq c|p_\varepsilon| + (|y_\varepsilon| + |y^d|) + \tilde{\beta} \|y_\varepsilon\|
\end{aligned}$$

providing boundedness of $\lambda_\varepsilon + \varepsilon \Delta p_\varepsilon$ in $H^{-1}(\Omega)$. Along a further subsequence, we obtain the weak convergence $\lambda_\varepsilon + \varepsilon \Delta p_\varepsilon \rightharpoonup \lambda^*$ in $H^{-1}(\Omega)$ which, in the limit, in particular yields

$$\langle -\mathbf{b} \cdot \nabla p^* + \tilde{p}^* + y^* - y^d - \tilde{\beta} \Delta y^* - \lambda^*, \varphi \rangle = 0$$

for all $\varphi \in H_0^1(\Omega)$. Consequently (6.12a) has to be satisfied. In addition, λ^* is the limit of λ_ε in the sense of distributions. This can be seen by testing (6.10a) with an arbitrary element $\varphi \in C_c^\infty(\Omega)$.

$$\begin{aligned}
\lim_{\varepsilon \to 0} \langle \lambda_\varepsilon, \varphi \rangle_{D'(\Omega), C_c^\infty(\Omega)} &= \lim_{\varepsilon \to 0} \langle -\varepsilon \mathbf{b} \cdot \nabla p_\varepsilon + \tilde{b}^0 p_\varepsilon + y_\varepsilon - y^d - \tilde{\beta} \Delta y_\varepsilon, \varphi \rangle \\
&= \lim_{\varepsilon \to 0} -\varepsilon (\Delta \varphi, p_\varepsilon) + \lim_{\varepsilon \to 0} \langle -\mathbf{b} \cdot \nabla p_\varepsilon + \tilde{b}^0 p_\varepsilon + y_\varepsilon - y^d - \tilde{\beta} \Delta y_\varepsilon, \varphi \rangle
\end{aligned}$$

Consequently the limit in the sense of distributions defined by $\tilde{\lambda}$ satisfies the identity

$$(\tilde{\lambda}, \varphi) = \langle -\mathbf{b} \cdot \nabla p^* + \tilde{p}^* + y^* - y^d - \tilde{\beta} \Delta y^*, \varphi \rangle$$

The density of $C_c^\infty(\Omega) \subset H_0^1(\Omega)$ provides $\tilde{\lambda} = \lambda^*$. Next we define $S = \{x \in \Omega : \xi^*(x) > 0\}$. In the proof of Theorem 6.2.3 an important intermediate result provides $(p_\varepsilon, \xi_\varepsilon)_{L^2(U)} = 0$ for all subsets $U \subset \Omega$. Consequently we obtain

$$\lim_{\varepsilon \to 0} (p_\varepsilon, \xi_\varepsilon)_{L^2(S)} = 0$$

but since p_ε and ξ_ε are only converging weak in $L^2(\Omega)$ we can not establish this equality for the $L^2(\Omega)$ product of the weak limits and obtain a limiting result only. To prove the sign condition of λ^* on the inactive set of y^* we introduce the following set, defined for an arbitrary $s \in \mathbb{R}$, $s > 0$.

$$\Omega_+^s := \{x \in \Omega : y^*(x) \geq s\}$$

Obviously, we have $\Omega_+ = \bigcup_{s>0} \Omega_+^s$. Given any $\tau > 0$ we find some $s_\tau > 0$ with

$$|\Omega_+ \backslash \Omega_+^{s_\tau}| \leq \tau/3. \tag{6.13}$$

This is a consequence of the σ-additivity of the Lebesgue-measure and the disjoint partition

$$\Omega_+ = \bigcup_{k=0}^{\infty} \{x \in \Omega : 1/k > y^*(x) \geq 1/(k+1)\}.$$

The Lebesgue measure of the right hand side is a monotonically increasing sequence which is bounded from above and consequently converges to the measure of Ω_+. Since $y_\varepsilon \to y^*$ in $L^2(\Omega)$, Theorem A.4.9 provides the existence of a further subsequence denoted again by ε such that $y_\varepsilon(x) \to y(x)$ almost everywhere in Ω. Now the Theorem of Egorov A.4.10 yields, for any $\tilde{\tau} > 0$, the existence of a measurable set $\Omega_{\tilde{\tau}}$ with $|\Omega\backslash\Omega_{\tilde{\tau}}| \leq \tilde{\tau}$ and $y_\varepsilon \to y^*$ uniformly almost everywhere on $\Omega_{\tilde{\tau}}$. Considering (6.13) and setting $\tilde{\tau} = \tau/3$ we find for $\tilde{\Omega}^\tau = (\Omega_+^{s_\tau} \cap \Omega_{\tau/3})$

$$|\Omega_+\backslash\tilde{\Omega}^\tau| = |\Omega_+\backslash\Omega_+^{s_\tau} \cup \Omega_+\backslash\Omega_{\tau/3}| \leq \tau/3 + \tau/3 = (2/3)\tau$$

The uniform convergence of y_ε on $\tilde{\Omega}^\tau$ implies the existence of some $\tilde{\varepsilon} > 0$ such that

$$y_\varepsilon(x) > 0 \text{ for all } x \in \tilde{\Omega}^\tau \text{ and } \varepsilon > \tilde{\varepsilon}$$

which clearly shows $\tilde{\Omega}^\tau \subset \{x \in \Omega : y_\varepsilon(x) > 0\}$ for all $\varepsilon > \tilde{\varepsilon}$. Next we consider a sequence of positive real numbers τ_ε with $\sum_{\varepsilon\geq\tilde{\varepsilon}} \tau_\varepsilon = \tau/3$. Since $(y_\varepsilon, u_\varepsilon, \xi_\varepsilon, p_\varepsilon, \lambda_\varepsilon)$ is $\mathcal{E}$-almost C-stationary for every $\varepsilon > 0$ we find for each of the τ_ε a set E_{τ_ε} with $|\{y_\varepsilon > 0\}\backslash E_{\tau_\varepsilon}| \leq \tau_\varepsilon$ and the known condition for parings of λ_ε and functions $v \in H_0^1(\Omega)$ vanishing almost everywhere on $\{x \in \Omega : y_\varepsilon(x) > 0\}\backslash E_{\tau_\varepsilon}$. Monotonicity of the Lebesgue measure yields

$$|\tilde{\Omega}^\tau\backslash E_{\tau_\varepsilon}| \leq |\{y_\varepsilon > 0\}\backslash E_{\tau_\varepsilon}|$$

for all $\varepsilon \geq \tilde{\varepsilon}$. Now we set $\Omega^\tau = \tilde{\Omega}^\tau \cap \bigcap\limits_{\varepsilon\geq\tilde{\varepsilon}} E_{\tau_\varepsilon}$. Recalling $|A \cap B| = |A| - |A\backslash B|$ with arbitrary sets $A, B \subset \Omega$ and using subadditivity of the Lebesgue measure we find

$$|\Omega_+\backslash\Omega^\tau| = |\Omega_+| - |\tilde{\Omega}^\tau \cap \bigcap_{\varepsilon\geq\tilde{\varepsilon}} E_{\tau_\varepsilon}| \leq \underbrace{|\Omega_+| - |\tilde{\Omega}^\tau|}_{|\Omega_+\backslash\tilde{\Omega}^\tau|} + \sum_{\varepsilon\geq\tilde{\varepsilon}} |\tilde{\Omega}^\tau\backslash E_{\tau_\varepsilon}| \leq (2/3)\tau + \sum_{\varepsilon\geq\tilde{\varepsilon}} \tau_\varepsilon = \tau$$

Without loss off generality we may assume that Ω^τ is closed. Otherwise, we construct the measurable set $\Omega^{\tau/2}$ and define Ω^τ as a closed subset of it satisfying $|\Omega^{\tau/2}\backslash\Omega^\tau| \leq \tau/2$. Such a subset exists by the approximation properties of Lebesgue measurable sets (see, e.g. [49]). We now obtain

$$\lim_{\varepsilon\to 0} \langle \lambda_\varepsilon + \varepsilon\Delta p_\varepsilon, \varphi\rangle = \langle \lambda^*, \varphi\rangle = 0$$

for any $\varphi \in \{v \in C_c^\infty(\Omega) | v(x) = 0 \text{ for all } x \in \Omega\backslash\Omega^\tau\}$ since $\Omega^\tau \subset E_{\tau_\varepsilon}$ for all $\varepsilon \geq \tilde{\varepsilon}$ implies $\langle \lambda_\varepsilon, \varphi\rangle = 0$ for all $\varepsilon \geq \tilde{\varepsilon}$ and $\varepsilon\langle \Delta p_\varepsilon, \varphi\rangle \to 0$ for $\varepsilon \to 0$ which proves (6.12f). □

In case of $Int(\Omega^\tau) \neq \emptyset$, we can establish an additional result concerning the multiplier λ^*. Consider the subset $\{v \in C_c^\infty(\Omega) | v(x) = 0 \text{ for all } x \in \Omega\backslash\Omega^\tau, v_{|Int(\Omega^\tau)} \in C_c^\infty(Int(\Omega^\tau))\}$ of the functions defined in (6.12f). By the definition, for any $v \in H_0^1(Int(\Omega^\tau))$ there exist a sequence $\varphi_n \in C_c^\infty(Int(\Omega^\tau))$ with $\varphi_n \to v$ in $H_0^1(Int(\Omega^\tau))$. Extending the elements of this approximating sequence and the limit element v by zero to the whole set Ω, we find

$$\langle \lambda^*, v\rangle = \lim_{n\to\infty} \langle \lambda^*, \varphi_n\rangle = 0$$

for all functions v defined on Ω and satisfying $v(x) = 0$ almost everywhere on $\Omega \backslash Int(\Omega^\tau)$ and $v_{|Int(\Omega^\tau)} \in H_0^1(Int(\Omega^\tau))$. Since the latter set of functions contains those elements of $H_0^1(Int(\Omega^\tau))$ that can be extended by zero to Ω such that the resulting functions are elements of $H_0^1(\Omega)$, (6.12f) can be strengthened to

$$\text{for all } \tau > 0 \; \exists \, \Omega^\tau : |\Omega_+ \backslash \Omega^\tau| \leq \tau \text{ s.t. } \forall v \in H_0^1(\Omega), \;\; v = 0 \text{ a.e. in } \Omega \backslash \Omega^\tau : \; \langle \lambda^*, v \rangle = 0.$$

However, the inequality $|\Omega_+ \backslash Int(\Omega^\tau)| \leq \tau$ can not be established to the best of our knowledge. The proof of the $\mathcal{E}$-almost property (6.12f) becomes significantly less complex if the sequence $(y_\varepsilon, u_\varepsilon, \xi_\varepsilon, p_\varepsilon, \lambda_\varepsilon)$ is almost C-stationary for every ε. Then we don't have to discuss the sets E_{τ_ε} and only use $\Omega^\tau = \tilde{\Omega}^\tau$. We have to point out, that we can not expect to obtain something better than an $\mathcal{E}$-almost condition. This stems from the fact, that for any better result the equality

$$\Omega_+ = \bigcap_\varepsilon \{y_\varepsilon > 0\}$$

would have to hold. This stabilizing behavior is highly unlikely since there could be a set $E \subset \{y_\varepsilon > 0\}$ for all ε but $y^*(x) = 0$ for almost every $x \in E$ by the pointwise convergence of y_ε to y^*.

We are fully aware of the fact, that the assumed boundedness properties of the sequence of solutions to the stationarity system is a critical point. However, Proposition 6.1.5 and the form of the objective ensures, that the desired boundedness is attained at least by a sequence of solutions to the Problems P^ε. Moreover, a usual assumption in the field of optimal control of elliptic variational inequalities (see [95]) is boundedness of control and slack function of the complementarity system in $L^2(\Omega)$ which is then, by regularity of the differential operator, equivalent to boundedness of the states in $H_0^1(\Omega)$. Thus the additional assumption is not to strong.

Discussion

Considering (6.8) and (6.12) we find almost the same results for the considered strategies while the same assumptions had to be made for obtaining them. The limiting behavior (6.12e) can not be realized on the hyperbolic level and consequently the vanishing viscosity approach yields a slightly stronger condition. Both stationarity conditions have to be understood as conditions based on the consistency proof of the corresponding approximating problems. First order necessary optimality conditions can be established for those auxiliary problems and so the limit of the corresponding stationary points respecting the assumptions on the convergence behavior clearly introduce some necessary optimality condition for the original problem. From a theoretical point of view both methods can be used for algorithmic purposes.

We will utilize the second method for the following reason. The class of problems of optimal control subject to elliptic variational inequalities is well understood and efficient numerical methods are available for the problems.

We point out, that (6.8) can be obtained without a coupling condition for the parameter of the auxiliary first order systems as used in [95, Assumption 3.3.1.], where

$$(\alpha \sqrt{\gamma})^{-1} \leq c \tag{6.14}$$

was necessary. In the cited reference, the condition was used to establish the following properties of the limiting sequences. First, the boundedness of p_γ in $H_0^1(\Omega)$ providing weak convergence in $H_0^1(\Omega)$ and strong convergence in $L^2(\Omega)$ along a subsequence was established using (6.14). Second, the product condition $(p, \xi)_{L^2(\omega)} = 0$ for all subsets $\omega \in \Omega$ was verified with it. Moreover, $\langle \lambda, p \rangle \leq 0$ could be obtained. All of these points can not be proven in case of a linear first order differential operator and hence the condition is not necessary.

Remark 6.2.1. *So far it is an open question, whether one can reduce the problem* $(\tilde{P})$ *to the case of control functions of the form* (6.4) *only. Although Example 6.2 provides the existence of feasible state-control pairings for the underlying variational inequality problem not admitting such a representation, the constructed pairing can not be optimal for any data* y^d *in* $(\tilde{P})$*. This holds because the choice* $\tilde{u} \equiv 0$ *performs best in the objective among all* $\tilde{u}$ *that are non-positive on the domain* $B_{r/2}(\hat{x})$ *in Example 6.2. Consequently, the best performing control would admit a representation as in* (6.4)*. If the reduction is possible, the original problem would be equivalent to solving*

$$\begin{array}{ll} min & \frac{1}{2}|w - y^d|^2 + \frac{\tilde{\beta}}{2}\|w\|^2 + \frac{\beta}{2}|A^0 w|^2 \\ s.t. & w \in \mathbf{K}, \end{array}$$

a problem, that is substantially simpler than the original posed one.

6.3 Algorithmic Treatment

Theorems 6.2.3 and 6.2.4 prove, under certain boundedness assumptions on the primal variables y, u and ξ, that stationary points of $(\tilde{P}^{\varepsilon}_{\gamma,\alpha,\kappa})$ satisfying (6.9) converge first to a stationary point of (P^{ε}) satisfying (6.10) and then to a point satisfying (6.12) with a structure which is similar to $\mathcal{E}$-almost W-stationarity for problem $(\tilde{P})$ although the product condition $\langle \lambda^*, y^* \rangle = 0$ is missing. The constructive nature of the proofs allows for the design of an algorithm with the same convergence properties.

6.3.1 The Algorithm

Following the results presented above, the algorithm mainly relies on a continuation technique of the involved parameters. We basically have two problem parameters which we are able to control. First, γ is the penalization factor in problem $(\tilde{P}^{\varepsilon}_{\gamma,\alpha,\kappa})$. As in [73, 95] we couple the parameters α and κ with γ according to the updating rules from Assumption 6.1 and thus we use

$$\kappa = \gamma^{-1/2} \text{ and } \alpha = \alpha_0(\gamma_0/\gamma)^{1/2}. \tag{6.15}$$

Note that for α a smaller exponent can be chosen but a faster decrease is not possible. The second parameter of interest is the viscosity parameter itself. Whenever a certain criterion is met, the viscosity parameter is lowered to a fraction of the current value. We will consider this parameter independent of γ since we could not obtain any coupled updating rules for ε and γ from the theoretical considerations and it has an important effect on the nature of the problem if the viscosity parameter changes. Moreover, this updating strategy reflects the proves of the Theorems 6.2.3 and 6.2.4.
Inspired by the strategy used for establishing the stationarity system (6.12), the overall algorithm for computing such points is given in Algorithm 3.

Algorithm 3 Outer Loop

DATA: $y_d, f, c, \bar{\lambda}, (\gamma, \alpha, \kappa) > 0, \beta_\gamma > 1, \varepsilon > 0, \beta_\varepsilon \in (0,1),\ \beta_\gamma > 1$
Initialize (y^0, u^0, ξ^0, r^0)
Repeat
  Repeat
    - Compute Stationary Point $(y^{k+1}, u^{k+1}, \xi^{k+1}, r^{k+1})$ of $(\tilde{P}^\varepsilon_{\gamma,\alpha,\kappa})$ with initial value (y^k, u^k, ξ^k, r^k) using a Semismooth Newton Method and nested grids
    - Update $\gamma^+ = \gamma\beta_\gamma$
    - Compute $\kappa(\gamma^+), \alpha(\gamma^+)$
  Until Refinement Criterion is met
Update $\varepsilon^+ = \varepsilon\beta_\varepsilon$
Until Stopping Criterion is met

Reformulating the Stationarity System with Complementarity Functions

The central part of the presented algorithm consists of computing a stationary point of the auxiliary problem $(\tilde{P}^\varepsilon_{\gamma,\alpha,\kappa})$. Since this has to be done for several pairings of parameters we are interested in an efficient manner. For fixed $\varepsilon > 0$ and $\gamma > 0$ we treat the optimality system (6.9) by a semismooth Newton method known to be efficient for this kind of problems (see [72]). Therefore we first have to reformulate the optimality system including complementarity conditions into a system of semismooth equations. It is well known, that

$$0 = a - \max\{0, a - cb\} \Leftrightarrow a \geq 0, b \geq 0, ab = 0$$

holds, i.e. $a - \max\{0, a - cb\}$ is a so called NCP-function whose roots are satisfying the complementarity system. Such functions are well studied in the context of state constrained optimization problems ([72, 73, 75]). We point out, that there is a whole zoo of NCP functions available (see e.g. [147]) but this particular choice yields a nonsmooth system of equations which can be shown to be Newton differentiable in the sense of Definition 2.4.1 for the present system. We reformulate (6.9d) and (6.9e) as

$$\begin{aligned} 0 &= \xi - \max\{0, \xi - c_\mu \mu\} \\ 0 &= r - \max\{0, r + c_r((\xi, y) - \alpha)\} \end{aligned}$$

Further, we eliminate μ by (6.9c), choose $c_\mu = \kappa$ and use (6.9b) to get rid of the dual quantity p which allows for iterating on primal variables (y, u, ξ) and a real number only. Consequently, we obtain the system of equations

$$F : H^1_0(\Omega) \times H^1_0(\Omega) \times L^2(\Omega) \times \mathbb{R} \to V^* \times L^2 \times \mathbb{R} \times V^*$$

defined by

$$F(y, u, \xi, r) = \begin{bmatrix} y - y_d - \tilde{\beta}\Delta y - (\bar{\lambda} - \gamma y)^+ + \beta(A^\varepsilon)^* u + r\xi \\ \kappa\xi + (ry - \beta u)^- \\ r - (r + c_r((\xi, y) - \alpha))^+ \\ A^\varepsilon y - u - \xi - f \end{bmatrix} \tag{6.16}$$

For this system we find the following result.

Lemma 6.3.1. *$F(y,u,\xi,r)$ as defined above is Newton differentiable in the sense of Definition 2.4.1.*

Proof. Almost identical to [73, Proposition 5.5] except the discussion for the first equation. For this equation, the pointwise max $()^+ : L^q(\Omega) \to L^2(\Omega)$ is Newton differentiable for all $q \geq 2$ ([72]) and since a two dimensional domain ensures the embedding $H_0^1(\Omega) \to L^q(\Omega)$ for all $2 \leq q < \infty$, the first equation is Newton differentiable for $u \in H_0^1(\Omega)$. Then any step computed from the semismooth Newton method provides this regularity for the iterates u^l if $u^0 \in H_0^1(\Omega)$ is initialized that way since δu always solves an elliptic PDE with corresponding right hand side in $H^{-1}(\Omega)$ ([157, section 22.5]). □

The generalized Newton step for the Relaxed Regularized Problem is the solution of the system

$$\begin{bmatrix} Id - \tilde{\beta}\Delta Id + \gamma\chi_y Id & \beta(A^\varepsilon)Id & rId & \xi Id \\ r\chi_I Id & -\beta\chi_I Id & \kappa Id & \chi_I y Id \\ -c_r\chi_r\xi Id & 0 & -c_r\chi_r y Id & (1-\chi_r)Id \\ A^\varepsilon Id & -Id & -Id & 0 \end{bmatrix} \begin{pmatrix} \delta y \\ \delta u \\ \delta\xi \\ \delta r \end{pmatrix} = -F(y,u,\xi,r) \quad (6.17)$$

Here Id represents the identity mapping in the corresponding function space. Considering the second line in (6.16) we find, that since y and u are elements of $H_0^1(\Omega)$ and this space is closed w.r.t. $(\cdot)^+$ (see [94]), any ξ satisfying $F(y,u,\xi,r) = 0$ is an element of this space as well. Even though this might not hold for any intermediate step of the solution process, the final regularity encourages us to discretize the quantity conforming to the more regular space $H_0^1(\Omega)$ than only to $L^2(\Omega)$.
According to Theorem 2.4.1 the semismooth Newton method converges superlinear, if the family of generalized derivatives has bounded inverse operators in a neighborhood of the solution and the method is initialized sufficiently close to it. For the presented examples invertibility was observed on the discrete level at all times for the chosen Newton derivative.

Discretization of the System and Reduction

The system (6.17) still has to be considered in the corresponding function spaces. For a numerical realization we have to discretize the problem and therefore choose standard $P1$ finite elements (see e.g. [66]). For the model domain $\Omega = (0,1)^2 \subset \mathbb{R}^2$ we utilize a uniform mesh size $h = 1/(N+1)$ where $N \in \mathbb{N}$ is the number of inner grid points per dimension. The discretized domain Ω_h is defined by the set of points

$$\{x_{i,j} = (ih, jh) : 1 \leq i,j \leq N\}.$$

This net of points introduces a set of triangles of similar size and shape which is the discretized domain Ω_h. The $P1$ conforming finite elements are given as the head functions $\varphi_k, k = 1, ..., N^2$ with nodal values

$$\varphi_k(x_{i,j}) = \delta_{k,(i+j)}$$

and linear interpolation on each triangle of the tiling Ω_h in $\mathbb{R}^2$. For these functions we compute the following matrices.

Definition 6.3.1. *Discretized Operator*

$$
\begin{array}{ll}
S_{i,j} = \int_\Omega \nabla\varphi_i \nabla\varphi_j dx & M^{\tilde{b}}_{i,j} = \int_\Omega (b^0 - .5\sum_{k=1}^n \mathbf{b}^k_{x_k})\varphi_i\varphi_j dx \\
MM_{i,j} = \int_\Omega \mathbf{b}\nabla\varphi_i\varphi_j dx & M^y = M * \chi^\gamma_y \\
M_{i,j} = \int_\Omega \varphi_i\varphi_j dx & M^u = M * \chi_u \\
M^b_{i,j} = \int_\Omega b^0\varphi_i\varphi_j dx & \tilde{M}^u = M - M^u
\end{array}
$$

Whenever $\mathbf{b}$ or b^0 are non-constant functions, we might replace them by their piecewise linearizations defined by the nodal values of the discretization. In the definition of the matrices above, we used the vectors χ^γ_y and χ_u which are given as

$$
(\chi^\gamma_y)_i = \begin{cases} 1 & \text{if } \bar\lambda_i - \gamma y_i \geq 0 \\ 0 & \text{else} \end{cases} \quad \text{and} \quad (\chi_u)_i = \begin{cases} 1 & \text{if } r y_i - \beta u_i \geq 0 \\ 0 & \text{else} \end{cases}
$$

This form was chosen, as it is quite common to express activity in the context of Primal Dual Active Set strategies by activity of the nodal values. Utilizing a finite difference discretization of (6.17) (see [66]) this choice is obvious and was used for example in [72, 73]. In general the set $\{x \in \Omega : \bar\lambda(x) - \gamma y(x) > 0\}$ does not admit its boundary on the points $x_{i,j}$ defining the discretized domain Ω_h. Consequently, the mass matrix M^y (and M^u respectively) would have to be computed in each step of the iteration requiring an analysis of each triangle $\mathcal{T} \in \Omega_h$. For a fixed iterate y the mass matrix including the exact active set is given as

$$
M^y_{i,j} = \int_\Omega \varphi_i(x)\varphi_j(x)\chi_{\{\bar\lambda(x) - \gamma y(x) > 0\}}.
$$

Thus the matrix M^y from Definition 6.3.1 introduces an additional error to the problem since it assumes for every element of the discretization Ω_h that it is completely inactive whenever a single base point of the discretization is inactive. However, this method of overestimating the active set is used frequently in the numerical treatment of variational inequalities even for a discretization based on finite elements as it can be seen for example in [17, 70, 127]. The error tends to zero with increasing fineness of the mesh.
Employing Definition 6.3.1 we are able to define the discretized counterpart of (6.17) as

$$
\begin{bmatrix}
M + \tilde\beta S + \gamma M^y & \beta(A^\varepsilon_a) & rM & M\xi \\
r(\tilde{M}^u) & -\beta(\tilde{M}^u) & \kappa M & \tilde{M}^u\mathbf{y} \\
c_r\chi_r\xi' M & 0 & c_r\chi_r\mathbf{y}' M & (1-\chi_r) \\
A^\varepsilon & -M & -M & 0
\end{bmatrix}
\begin{pmatrix} \delta\mathbf{y} \\ \delta\mathbf{u} \\ \delta\xi \\ \delta\mathbf{r} \end{pmatrix}
= -F(\mathbf{y}, \mathbf{u}, \xi, r) \qquad (6.18)
$$

which is a linear system of $3N^2 + 1$ unknown variables. In order to reduce the computational effort which has to be invested for solving this system, we note, that $\delta\xi$ only occurs as matrix vector product with the mass matrix M. Consequently, we can reduce the total system utilizing its second row of blocks and substitute

$$
\begin{aligned}
M\delta\xi = & \frac{1}{\kappa}[r(M^u - M)\delta\mathbf{y} + \beta(M - M^u)\delta\mathbf{u} \\
& + (M^u - M)\mathbf{y}\delta r - M(\kappa\xi - \beta\mathbf{u} + r\mathbf{y}) + M^u(r\mathbf{y} - \beta\mathbf{u})]. \qquad (6.19)
\end{aligned}
$$

The resulting reduced system only contains $2N^2 + 1$ unknowns and the remaining vector $\delta\xi$ can be obtained from (6.19) by solving a linear system with N^2 variables. Thus the iterates of the semismooth Newton method are computed from the following system.

$$\begin{bmatrix} M+\tilde{\beta}S+\gamma M^y-\frac{r^2}{\kappa}\tilde{M}^u & \beta(A_a^\varepsilon+\frac{r}{\kappa}\tilde{M}^u) & M\xi-\frac{r}{\kappa}\tilde{M}^u\mathbf{y} \\ \chi_r c_r(\xi' M-\frac{r}{\kappa}\mathbf{y}'\tilde{M}^u) & \chi_r c_r\frac{\beta}{\kappa}\mathbf{y}'\tilde{M}^u & (1-\chi_r(1+\frac{c_r}{\kappa}\mathbf{y}'\tilde{M}^u\mathbf{y})) \\ A^\varepsilon+\frac{r}{\kappa}(\tilde{M}^u) & -(M+\frac{\beta}{\kappa}\tilde{M}^u) & \frac{1}{\kappa}\tilde{M}^u\mathbf{y} \end{bmatrix} \begin{pmatrix} \delta\mathbf{y} \\ \delta\mathbf{u} \\ \delta\mathbf{r} \end{pmatrix}$$
$$= \begin{bmatrix} M\mathbf{y}_d+M^y\bar{\lambda}-(M+\gamma M^y+\tilde{\beta}S-\frac{r^2}{\kappa}\tilde{M}^u)\mathbf{y}-(\beta A_a^\varepsilon+\frac{\beta r}{\kappa}\tilde{M}^u)\mathbf{u} \\ (\chi_r-1)r-c_r\alpha-\chi_r\frac{c_r}{\kappa}\mathbf{y}\tilde{M}^u(r\mathbf{y}-\beta\mathbf{u}) \\ -(A^\varepsilon+\frac{r}{\kappa}\tilde{M}^u)\mathbf{y}+(M+\frac{\beta}{\kappa}\tilde{M}^u)\mathbf{u}-M\mathbf{f} \end{bmatrix} \tag{6.20}$$

A further reduction is possible in the case $\chi_r = 0$. Obviously, the second row of (6.20) reduces to $\delta r = r - c_r\alpha$ leaving a problem of $2N^2$ variables. Any further reduction would only be possible, when a mass lumping technique is applied in Definition 6.3.1 replacing the five-band matrix M (with **full** inverse) by a diagonal matrix (with diagonal inverse) according to some lumping scheme as Row-Sum lumping (see for example [66]). In this case we could use the third row of blocks in (6.20) to substitute $\delta\mathbf{u}$ from the system and end with dimension N^2+1. Since mass lumping introduces further numerical errors and the numerical examples worked out sufficiently fast, we decided against the lumping for this problem and use (6.20).

Note, that the reduction step described above is highly recommended. Since the discretization of the differential operator yields a nonsymmetric matrix, the whole linear system is nonsymmetric. Consequently, the computational effort for solving the system is considerably large and any possibility of saving degrees of freedom in the problem significantly pays off in the sense of computation time needed to solve the linear system.

Globalization

A possible way for the local convergence behavior of the semismooth Newton method is a globalization based on backtracking along the so called Newton path τ. In case of differentiable functions, this path is the line segment $[x, x+d_N]$ for the descent direction d_N obtained as solution of the linearized system. In the context of semismooth Newton methods, a descent property along a suitable chosen nonlinear path has been introduced for example in [131] but the method is expensive with respect to computational effort. Instead of backtracking along the nonlinear path we decided to use an Armijo type line search along the linear path

$$p_x(\tau) = x + \tau d_N$$

for the semismooth Newton direction d_N. Although heuristic, this technique has proven to be efficient concerning numerical performance in (see [71, 73, 95]). The algorithmic procedure is presented in Algorithm 4.

Algorithm 4 Globalization of Semismooth Newton Method

DATA: $\mathbf{x} = (\mathbf{y}, \mathbf{u}, \xi, r)$ and $\delta\mathbf{x} = (\delta\mathbf{y}, \delta\mathbf{u}, \delta\xi, \delta r)$ solving (6.18), $\sigma \in (0,1)$, $\tau = 1$, $\delta\tau \in (0,1)$
$l = 0$
While (RESIDUAL$(\mathbf{x}+\tau\delta\mathbf{x}) \geq (1-\sigma\tau)$RESIDUAL$(\mathbf{x})$) and $(l \leq l_{MAX})$
    $\tau = \tau \cdot \delta\tau$
    $l = l + 1$
$\mathbf{x}_+ = \mathbf{x} + \tau\delta\mathbf{x}$

Basically this strategy ensures a decrease in the residual of the nonsmooth system in every iterate. In the numerical tests, this strategy did not fail and in none of the considered problems the procedure terminated because l_{MAX} was exceeded. For the computation of the RESIDUAL we recall the image space of F to be

$$H^{-1}(\Omega) \times L^2(\Omega) \times \mathbb{R} \times H^{-1}(\Omega).$$

Let F be given as the vector $(\mathcal{R}_1, \mathcal{R}_2, \mathcal{R}_3, \mathcal{R}_4)$ each living in the corresponding part of the image space. The norm of the $L^2(\Omega)$ and the $\mathbb{R}$ part of F, $\mathcal{R}_2, \mathcal{R}_3$ can be evaluate efficiently by the mass matrix M and the absolute value. Formally, the first and last part of F, $\mathcal{R}_1, \mathcal{R}_4$ each of length N^2, are elements of $H^{-1}(\Omega)$ since they represent the adjoint and primal equation. To evaluate the corresponding norm, we use the Riesz representation Theorem (see [154]) and compute

$$S\hat{\mathcal{R}}_i = \mathcal{R}_i$$

for $i = 1, 4$ and the stiffness matrix S. We now obtain the following formula for the residual of the system.

$$\text{RESIDUAL} = (\hat{\mathcal{R}}_1^\top S \hat{\mathcal{R}}_1)^{1/2} + (\mathcal{R}_2^\top M \mathcal{R}_2)^{1/2} + |\mathcal{R}_3| + (\hat{\mathcal{R}}_4^\top S \hat{\mathcal{R}}_4)^{1/2} \tag{6.21}$$

We use the classical Backtracking with $\delta\tau = 0.5$. For the value of σ we follow the recommendation in [121] and choose $\sigma = 10^{-4}$. Note that the line search performed well independent of the choice of this value in the numerical experiments. We could not observe significant change in the number line search iterations when varying this number.

Initialization

Although we introduced a globalization strategy for the semismooth Newton method in Algorithm 4, we would like to obtain a starting point being at least close to the zone of attraction, were the method converges superlinearily.

For the initialization of the algorithm several approaches are possible. A natural choice would be the discrete counterpart of the overall feasible point $(0, -f, \xi)$ already utilized in the proofs ensuring existence of solutions to the penalized regularized problems, and $r = 0$.

Since we do not expect the examples to have such a solution, this choice might still be far away from the solution. We therefore decided to use a starting point close to the problem in the following way. We consider the optimization problem

$$\begin{array}{ll} \min & |y - y^d|^2 + \frac{\tilde{\beta}}{2}\|y\|^2 + \frac{\beta}{2}|u|^2 + \frac{1}{2\gamma}|(\bar{\lambda} - \gamma y)^+|^2 + \frac{\kappa}{2}|\xi|^2 \\ \text{s.t.} & -\varepsilon\Delta y + \mathbf{b} \cdot \nabla y + b^0 y - f - u = \xi \\ & \xi \geq 0 \end{array}$$

In contrast to $(\tilde{P}^{\varepsilon}_{\gamma,\alpha,\kappa})$, only the condition on (y, ξ) was dropped creating a neighboring problem. We expect the solution (y^0, u^0, ξ^0) to be a good initial guess for the algorithm. Note that this is completely heuristic and can not be supported by mathematical arguments. The auxiliary problem is solved by the semismooth Newton method as well. For the initialization of the real number, we have chosen $r^{(0)} = 0$.

As outlined in Algorithm 3, the semismooth Newton method, used to determine the solution of the nonlinear first order system (6.9) for different values of γ and ε, is initialized in every iteration by the stationary point of the preceding problem. The impact of the solutions to the preceding problems as initialization strongly depends on the update strategy of the parameter γ and later ε. If γ is increased to aggressively, two adjacent problems might not have much in common providing a poor initial guess. This directly requires more semismooth Newton steps to solve the nonlinear system. Conversely, a mild or conservative strategy for γ would need more steps in the outer algorithm while each of the nonlinear problems can be solved much more efficiently.

Updating, Mesh-Refinement and Stopping Criteria

First we point out, that we have three parameters which we have to take care of. The parameter γ which affects the penalization and regularization in the approximating problems $(\tilde{P}^{\varepsilon}_{\gamma,\alpha,\kappa})$, the mesh width h used for the discretization of the problems and the viscosity parameter ε itself. The derivation of the stationarity system (6.12) in particular relies on the fact, that we actually have stationary points of the problems (P^{ε}). Thus, the updating of ε will not be coupled to the parameter γ directly and therefore only occurs in the most outer loop of Algorithm 3.

We will now define the numerical parameters the algorithm from the in- to the outside. Here, the inner loop is determining the solution of the nonlinear stationarity system. The updating of γ is referred to as middle loop. Finally, the updating of ε will be done in the outer loop.

The main part of the presented algorithm consists of solving the nonlinear system (6.9) for certain parameters $(\gamma, \alpha, \kappa, \varepsilon)$. This will be done on a sequence of nested grids with mesh sizes $\{h_i\}$ for $1 \leq i \leq 5$ and $h_i = 2^{-(3+i)}$. In each inner iteration, the nonlinear system is solved by the semismooth Newton method with a globalization according to Algorithm 4 with parameters highlighted in the corresponding section. The method terminates when the residual of the system, (6.21), is below the value $TOL_{Newton} = 10^{-10}$. This decision has been made as the numerical examples showed, that on the very coarse meshes up to h_3 very few Newton iterations are necessary to obtain the given tolerance (at most 4 Newton steps) and on the finer ones the high accuracy together with the conservative update strategy pays off as the globalization of the method almost never has to be used. Since the algorithm is well defined in the corresponding Hilbert spaces and the chosen discretization with continuous $P1$ finite elements is conforming, the convergence properties of it are passed to the discrete problems providing overall convergence if the requirements on boundedness are met.

As we utilize a hierarchy of meshes for the discretization of the problem, we have to specify when we have to change from the current to a finer mesh. According to heuristic considerations in [73, 95] we will couple this to the parameter γ. The mesh is refined whenever

$$\gamma \geq c_{grid} h^{-4} \tag{6.22}$$

holds. Note that c_{grid} should be chosen small enough such that it does not influence the order of h on the right hand side. If the constant is chosen to large, the discretization error for the final iterates on one mesh is very large resulting in several additional Newton steps after refinement to regain accuracy with respect to the current discretization. The vectors $\mathbf{y}$ and $\mathbf{u}$ only defined on the interior nodes of the discretization of Ω are discrete representations of $H_0^1(\Omega)$ functions. The regularity of u stems from the fact, that for every $\varepsilon > 0$ the adjoint state p is an element of $H_0^1(\Omega)$ and by the coupling (6.9b) the same holds true for the control. When the mesh is refined we have to prolongate the discrete representations to the finer mesh.

For this task a linear interpolation scheme is utilized which is available as built in function of Matlab.

We point out the following important property of (6.22). When only considering the primal equation (6.9f), the constant c_{grid} grows even for linear partial differential equations in the same order as ε decreases (see [23]). Thus, the discretization error would grow in the outer loop of the algorithm forcing us to refine the mesh in this loop as well. y_ε solves in addition to the primal equation with degenerating differential operator the adjoint equation (6.9a) as well, we assume for this equation a discretization error of

$$|y_\varepsilon - y_\varepsilon^h| \leq c(\tilde{\beta})h\|y_\varepsilon\|$$

as it can be found in [68, chapter 8.4] for H^2 regular second order problems. Since the terms in the adjoint equation remain bounded in $H^{-1}(\Omega)$ if the assumptions of Theorem 6.2.4 are met. Here the constant c depends on the parameter $\tilde{\beta}$ does not change in the convergence process.

While providing a very useful tool for the theoretical consideration of problems of optimal control subject to state constraints, the penalization technique utilized in this work carries a certain disadvantage for the numerical realization. Although solutions of the approximating problems $(\tilde{P}^\varepsilon_{\gamma,\alpha,\kappa})$ converge to a solution of (P^ε) for $\gamma \to \infty$, we can not expect them to fulfill the nonegativity condition $y_\gamma \geq 0$ to hold for finite γ. Thus the constraint will be violated by all of the solutions to the stationarity systems (6.9) and any stopping criterion based on measuring complementarity fails. In [95] this was overcome by letting the algorithm terminate as soon as γ exceeds $c_{grid}h^{-4}$ for the smallest mesh width considered. We will use a similar condition for the termination of the middle loop and start the ε updating. In addition we will consider a certain extended residual of the system (6.9) to measure the quality of the stationary points. This quantity is an extension of the residual of the system, RESIDUAL. In addition it measures the violation of (6.10f), (6.10g) and (6.11) and is defined as

$$\text{RESIDUAL}_{C-Stat} = \text{RESIDUAL} + |\lambda^+| + |\lambda'\mathbf{y}| + |\beta\mathbf{u}'\lambda|$$

with $\lambda = M((\bar{\lambda} - \gamma\mathbf{y})^+ - r\xi)$. We will present the development of this value along the steps of the algorithm.

In the outer loop, ε will be decreased until the viscosity parameter falls below a certain value TOL_ε. Thus, the overall stopping criterion for the algorithm is $\varepsilon \leq TOL_\varepsilon$.

6.3.2 Examples

The numerical experiments suggested, that most of the difficulties stem from the data itself. The following three examples are problems, where the above algorithm yields satisfactory results. Let the domain $\Omega = (0,1) \times (0,1)$ be given. We will consider the following problem of optimal control

$$\begin{array}{lll} \min\limits_{(y,u)} & \frac{c}{2}|y - y^d|^2_{L^2(\Omega)} + \frac{\tilde{\beta}}{2}\|y\|^2_{H^1_0(\Omega)} + \frac{\beta}{2}|u|^2_{L^2(\Omega)} & \\ \text{s.t.} & y \in \{v \in H^1_0(\Omega) | v \geq 0 \text{ a.e. }\} & = \mathbf{K} \\ & \langle \mathbf{b}\cdot\nabla y + b^0 y - f - u, v - y\rangle & \geq 0 \ \ \forall v \in \mathbf{K} \end{array}$$

for different values of c and $\tilde{\beta} = 10^{-2}, \beta = 10^{-3}$ fixed. The first order operator is defined by

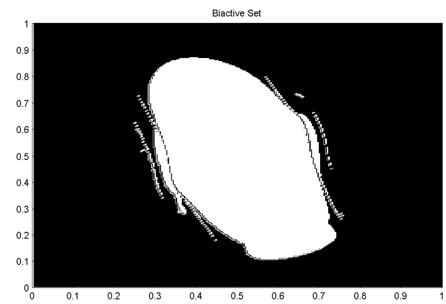
$$\mathbf{b} = (-2, 5)^\top \text{ and } b^0 = 0.1.$$

Finally, the distributed force f will be set to zero. For the algorithmic treatment we have initialized $\varepsilon = 0.01$ as small from the beginning. α_0 is chosen to be 1. In each iteration of the middle loop, γ will be increased conservatively by multiplying with $\beta_\gamma = 1.5$. This value has been chosen as the initial value of ε is already small and we wanted to avoid difficulties in the γ update phase due to this choice. The algorithm terminates if the viscosity parameter goes below $TOL_\varepsilon = 10^{-7}$. For the first and the third examples , ε will be decreased in the outer loop by multiplying with $\beta_\varepsilon = 0.75$. According to this choice, the outer loop will have $ceil(\ln(10^{-5})/\ln(0.75)) = 41$ iterations for this examples. The second example has sufficiently been solved for $\beta_\varepsilon = .85$. As in [95] we have chosen $c_{grid} = 1$. For each of the following examples you will find surface plots of the primal variables y_ε and u_ε as well as the multiplier $\lambda_\varepsilon = (y_\varepsilon)^+ + r_\varepsilon \xi_\varepsilon + \beta\varepsilon\Delta u_\varepsilon$ for the first and the last step of the outer loop. We restrict ourselves on plotting the interior nodes since all depicted variables have zero boundary conditions.

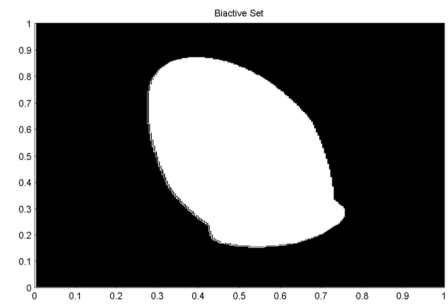
Example 6.3. *In the first example we use*

$$y^d(x_1, x_2) = \eta(x_1) \cdot \eta(x_2)$$

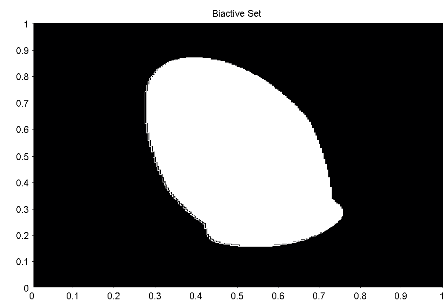
with $\eta(x) = 10^7 e^{-1/(0.25^2-(x-0.5)^2)}$ *if* $|x - 0.5| < 0.25$ *and* 0 *else and* $c = 100$. *The example was chosen as it contains a significant biactive set for all* $\varepsilon > 0$. *We have depicted these sets for several steps of the algorithm in black in Figure 6.1.*



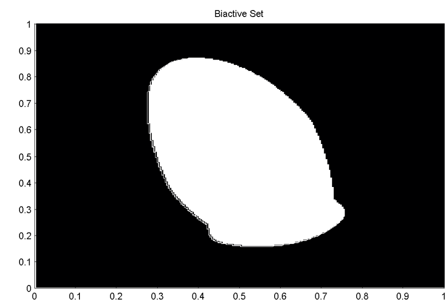
(a) Biactive Set Step 1



(b) Biactive Set Step 17



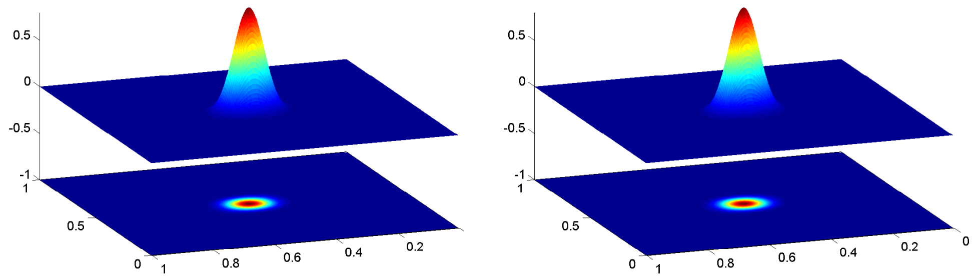
(c) Biactive Set Step 33



(d) Biactive Set Step 41

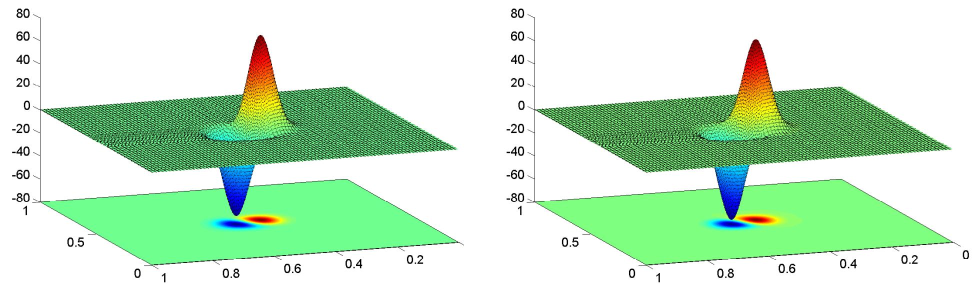
Figure 6.1: Biactive Set for different values of ε depicted in black

In Figure 6.2 we have depicted the state of the system for the first and final viscosity parameter when the middle loop has terminated. Figure 6.3 shows the controls for the same stages of the algorithm and Figure 6.4 displays the multipliers. Figure 6.5 indicates, that the norms of the iterates stay bounded while ε decreases.



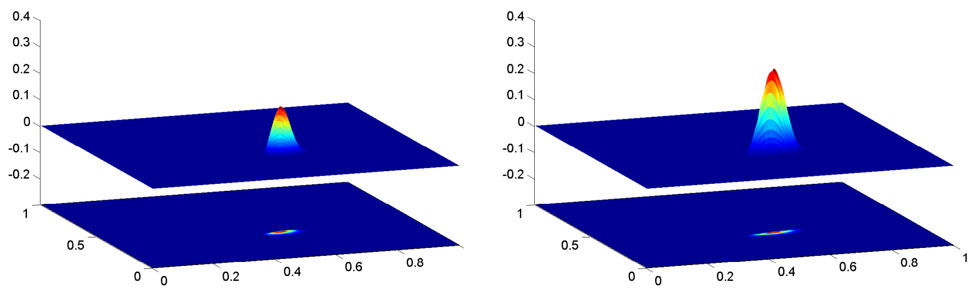
(a) Step 1 (b) Step 41

Figure 6.2: States of Example 6.3



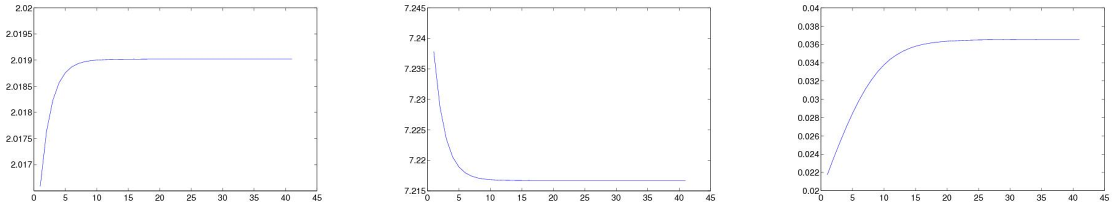
(a) Step 1 (b) Step 41

Figure 6.3: Control in Example 6.3



(a) Step 1 (b) Step 41

Figure 6.4: Multiplier of Example 6.3



(a) $\|y_\varepsilon\|$ (b) $|u_\varepsilon|$ (c) $|\xi_\varepsilon|$

Figure 6.5: Behavior of the norms under refinement of ε in Example 6.3

Example 6.4. *In this example we have chosen data which do not vanish identically on the boundary. The data function is given as*

$$y^d(x_1, x_2) = \max(0, \cos(15\sqrt{x_1^2 + x_2^2}))$$

and depicted in Figure 6.6.

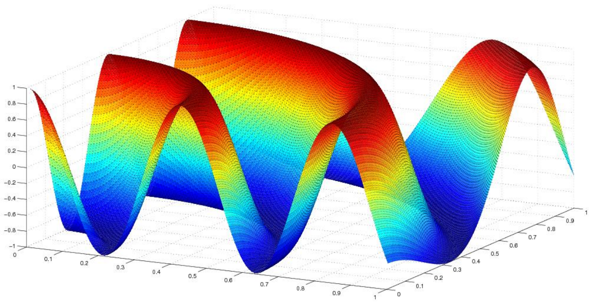
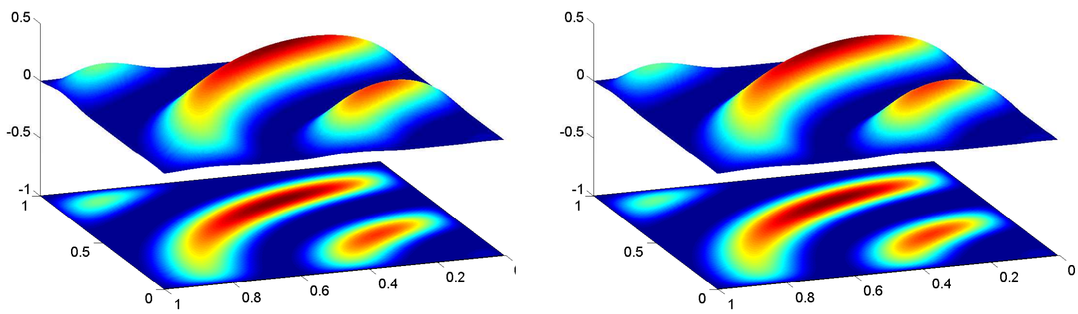


Figure 6.6: Data of Example 6.4

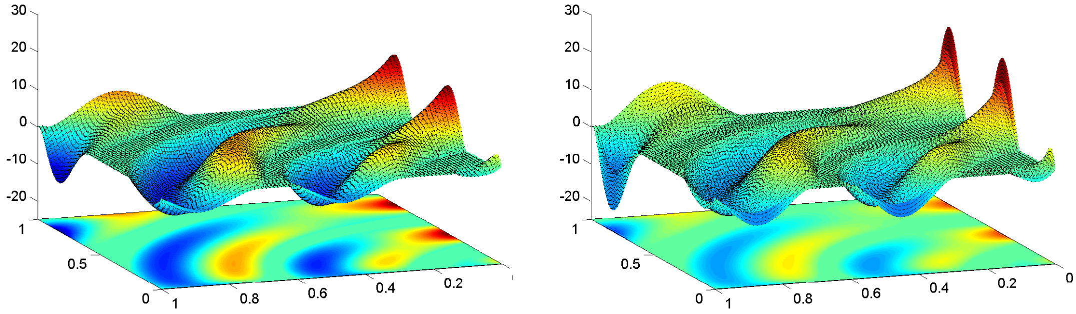
In this example we used $c = 1$*. In Figure 6.7 we have depicted the state of the system for the first and final viscosity parameter when the middle loop has ended.*



(a) Step 1 (b) Step 41

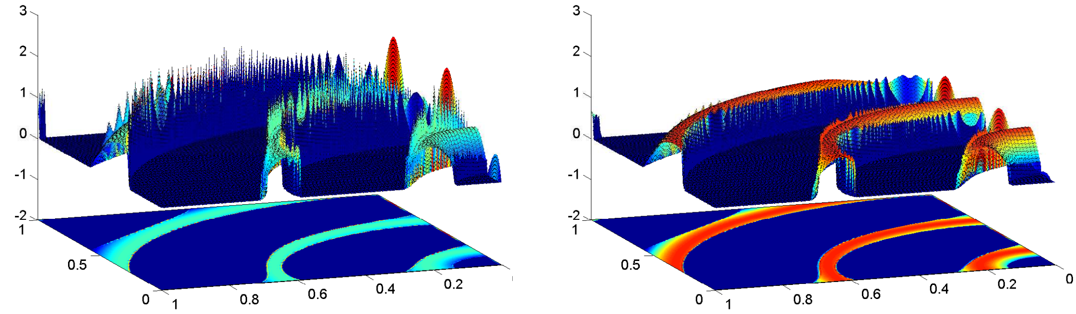
Figure 6.7: States of Example 6.4

Figure 6.8 shows the controls for the same stages of the algorithm and Figure 6.9 displays the



(a) Step 1 (b) Step 41

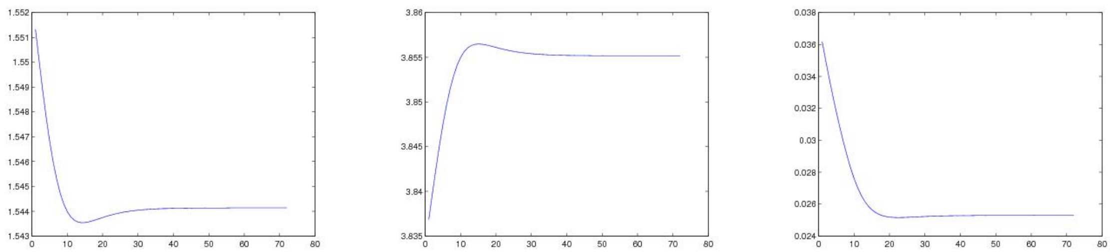
Figure 6.8: Control in Example 6.4



(a) Step 1 (b) Step 41

Figure 6.9: Multiplier of Example 6.4

multipliers. Again, Figure 6.10 indicates, that the norms of the iterates remain bounded under when ε decreases.



(a) $\|y_\varepsilon\|$ (b) $|u_\varepsilon|$ (c) $|\xi_\varepsilon|$

Figure 6.10: Behavior of the norms under refinement of ε in Example 6.4

Example 6.5. *In this example we have chosen discontinuous data as*

$$y^d(x_1,x_2) = \begin{cases} 0.5 & \textit{if } x_2 \geq 0.75 \\ 0.5 & \textit{if } x_2 \in (0.5, 0.75), x_1 \in (0, 0.25) \cup (0.75, 1) \\ 0.5 & \textit{if } x_2 \in (0.25, 0.5), x_1 \in (0.25, 0.75) \\ 0 & \textit{if } x_2 \in (0.25, 0.5), x_1 \in (0, 0.25) \cup (0.75, 1) \\ -0.25 & \textit{else} \end{cases}$$

and can be seen in Figure 6.11. Here we have also chosen $c = 1$ and were in particular interested whether the feature in the middle is detected.

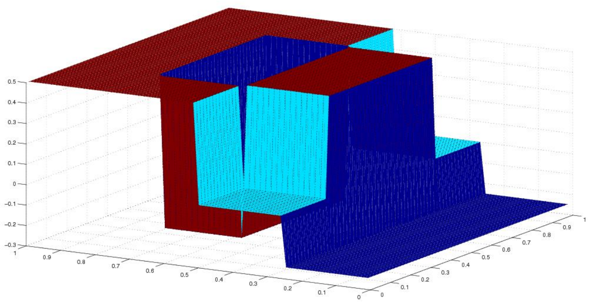
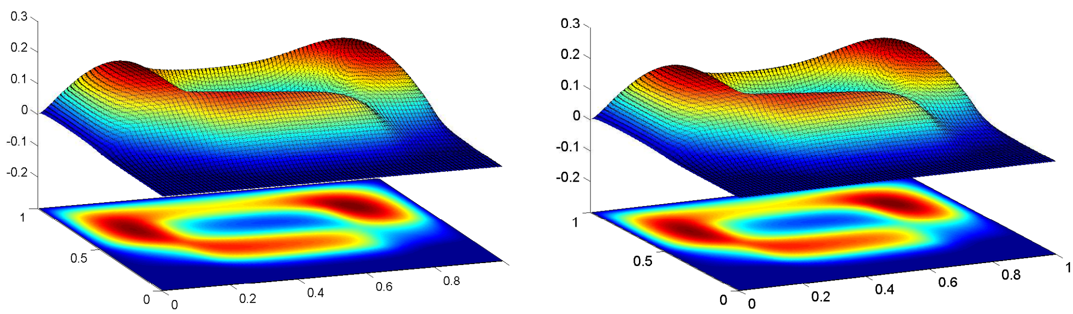


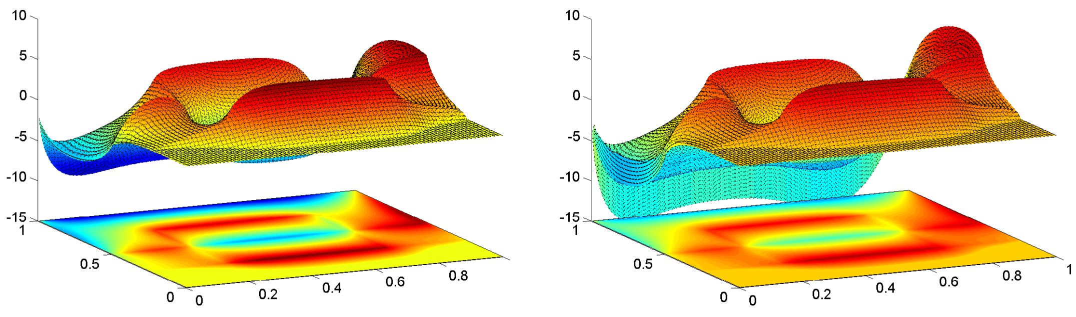
Figure 6.11: Data of Example 6.5

In Figure 6.12 we have depicted the state of the system for the first and final viscosity parameter when the middle loop has ended. Figure 6.13 and 6.14 present the corresponding controls and multipliers respectively.



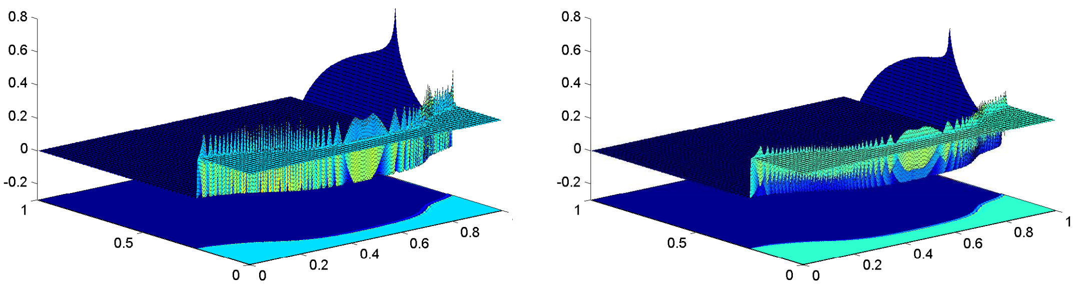
(a) Step 1 (b) Step 41

Figure 6.12: States of Example 6.5



(a) Step 1 (b) Step 41

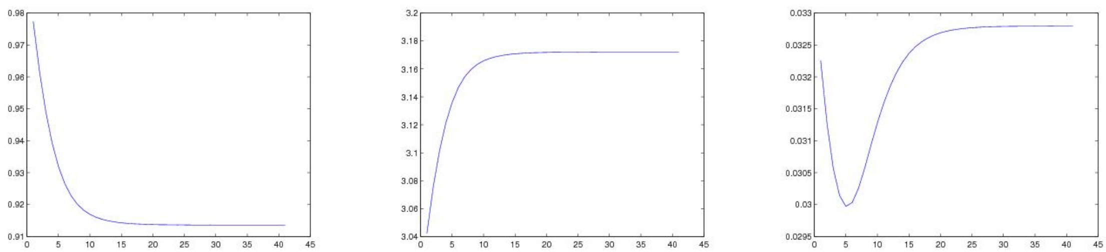
Figure 6.13: Control in Example 6.5



(a) Step 1 (b) Step 41

Figure 6.14: Multiplier of Example 6.5

Finally Figure 6.15 presents the norm of the iterates at all stages of the outer loop.

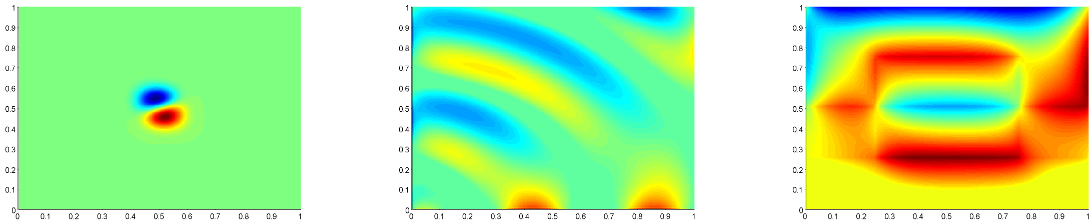


(a) $\|y_\varepsilon\|$ (b) $|u_\varepsilon|$ (c) $|\xi_\varepsilon|$

Figure 6.15: Behavior of the norms under refinement of ε in Example 6.5

Discussion of the Examples

For low viscosity parameters, the underlying partial differential equation can be interpreted as a transport equation in a steady state. The vector $\mathbf{b}$ defines the direction of transport. In this setting, the control u describes where material is allocated ore removed respectively. Figure 6.16 provides a top view of the control functions in the presented examples for the smallest value of ε achieved by the algorithm.



(a) Example 6.3 (b) Example 6.4 (c) Example 6.5

Figure 6.16: Top View of Control Functions u_ε in the examples

The transport behavior is clearly evident if one compares to the final states. Red represents the areas were material is allocated and blue where it is removed. We point out Figure 6.16c where the control has to act strongly near the upper boundary since material is transported in this direction and zero boundary conditions have to be met. In Figure 6.16a the vector **b** can be observed best.

Numerical Performance

The numerical computations were performed on the computer cluster of the Humboldt-Universität zu Berlin with Dual Xeon Quad Core nodes and 48 GB memory. The code was written in Matlab and the linear systems solved with the built-in function mldivide.
Consistency:
We have tested the algorithm on chosen examples of [73] with the middle loop only and reconstructed the results.
Superlinear Convergence:
In Figure 6.17 we depicted the ratios

$$\rho_k = |y_\gamma^{k+1} - y_\gamma^*|/|y_\gamma^k - y_\gamma^*|$$

for the involved variables $(y_\gamma, u_\gamma, \xi_\gamma, r_\gamma)$ of the semismooth Newton method at the 44th step of the middle loop in Example 6.5 where we have $\varepsilon = \varepsilon_0$, $\gamma = 1.105710^7$ while κ and α are determined according to (6.15). The values of $(y_\gamma^*, u_\gamma^*, \xi_\gamma^*, r_\gamma^*)$ were obtained by approximating the stationary point of this particular problem $(\widetilde{P}^\varepsilon_{\gamma,\alpha,\kappa})$ with higher accuracy, i.e. the nonlinear system (6.16) was solved with $TOL_{Newton} = 10^{-12}$ for this particular problem. The difference in the state y_γ and control u_γ were measured in the discrete $H_0^1(\Omega)$ norm while we took the discrete $L^2(\Omega)$ norm for the difference with respect to ξ_γ and the absolute value for r_γ. The locally superlinear convergence of the semismooth Newton method is indicated in Figure 6.17 and a similar convergence behavior was observed in all stages of the middle and outer loop in all of the presented examples. The convergence properties of the inner loop were analyzed in detail in [95] and thus we will not further discuss it in this work.
Iterations of the middle loop:
In the Table 6.1 we present the iteration numbers of the semismooth Newton method for the initial γ update strategy. For the middle loop we only present the total amount of Newton steps

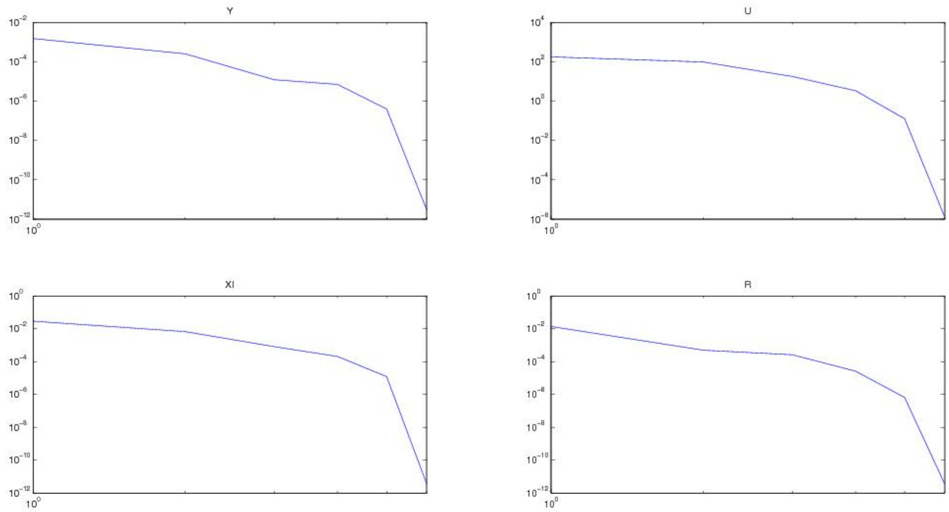


Figure 6.17: Convergence Plot of the Iterates

per mesh and not for every value of γ. In the outer loop we made the following observation

	Example 6.3	Example 6.4	Example 6.5
$h = 2^{-4}$	44	51	49
$h = 2^{-5}$	31	26	28
$h = 2^{-6}$	22	25	29
$h = 2^{-7}$	22	20	27
$h = 2^{-8}$	24	25	33

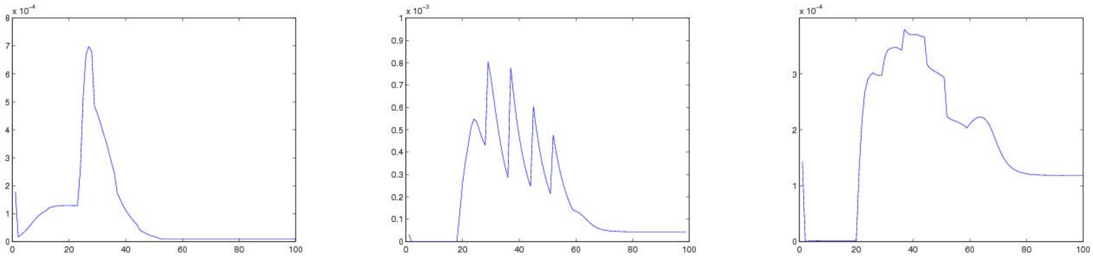
Table 6.1: Iteration counts of SSN for different parameters

for the iteration count of the Newton method. Either the number of steps was really small (at most 4 iterations directly after starting the outer loop) and then going down to 1 quickly or there was no convergence at all. The latter point will be addressed below.

Residual of C stationarity:

Figure 6.18 provides the value of RESIDUAL_{C-Stat} varying over the middle steps of the algorithm for the three presented examples. These computations are made for $\beta_\gamma = 1.25$ and stopped after 100 iterations of the outer loop to obtain comparable results

For Example 6.3 and 6.5 the scale goes up to 10^{-3} and for Example 6.4 up to 10^{-4}. In the first phase for small values of γ and on coarse meshes, the residual is very small although the problem itself is a problem of optimal control subject to a partial differential equation rather than subject to a variational inequality. In Figure 6.18a and 6.18c we observe that for γ sufficiently large we get drastically better whenever passing to a finer mesh and mildly improve for fixed mesh size. Figure 6.18b does not fit into this characteristics and is larger by a whole order of magnitude. This is an indicator, that stopping criteria based on this residual are not promising. Thus we have not further tried to use it for steering the algorithm.



(a) Example 6.3 (b) Example 6.4 (c) Example 6.5

Figure 6.18: Residual of C Stationarity

Observations and Limitations

Concerning Example 6.4 we point out, that we had to use a smaller updating parameter for the value of the viscosity than in the other presented problems. With the same value for β_ε as in the other examples the algorithm terminates after 2 steps of the outer loop failing to converge even if we allow more than 100 Newton steps.
The same holds true for Example 6.4 and 6.5 if the regularization parameter $\tilde{\beta}$ is chosen to small. Testing with $\tilde{\beta} = 10^{-3}$ both problems failed to converge even below $\varepsilon = 10^{-4}$. We believe, that this failure of convergence is because of the following reason.
From Theorem 6.2.4 we know, that the active set of stationary points for different values of ε does not stabilize. This can only be expected for the inactive set and is used for the construction of the $\mathcal{E}$ almost sets. Changing ε significantly influences the operator and thus the whole problem including the active sets on the discrete level. If the solution of the MPCC for the preceding value of ε is used as initialization for the current problem and the active set changes drastically, the resulting problem is almost as hard to solve as discrete variational inequalities since we have already increased the penalization-regularization parameter γ to a level where we expect the active set of the elliptic problem to be fixed. This idea is backed by the behavior of the residual which, although finding a direction of descent in any iteration, goes back to the order of the previous iteration if the active and inactive sets are identified at the new iterate. We point out that this is a problem specific issue. In Example 6.3 can we can set $\beta_\varepsilon = 0.1$ and chose any of the suggested $\tilde{\beta}$ and the algorithm converges.
To overcome this issue we have tested several approaches. First and most natural, a more conservative updating rule for ε can be used. For $\tilde{\beta} = 10^{-2}$ we demonstrated this strategy in Example 6.4. Testing both examples with $\tilde{\beta} = 10^{-3}$, $\beta_\varepsilon = 0.99$ was still to large to lower the viscosity parameter in the outer loop. The experiments indicate, that there are ranges of viscosity parameters, where the active set does not change to much and therefore large steps in ε are possible while in other ranges very little progress can be made.
A second possibility is to inflate the feasible set of the optimization problem again each time ε is decreased. As a consequence we have to invest additional steps of the middle loop for each level of ε. Again, experiments suggest that this update rule might work but we were not able to estimate an value how to decrease γ. For example this strategy worked for Example 6.4 with $\tilde{\beta} = 10^{-3}$ and $\gamma^+ = \gamma/\beta_\gamma^4$, making 4 iterations of the middle loop after updating ε, but failed for γ/β_γ^3. In both cases we have chosen $\beta_\varepsilon = 0.95$. This second method is interesting since it utilizes the relaxation-penalization approach for the MPCC again. So far we have only used the method as a very efficient way to obtain a sufficiently good initialization for the

outer loop. Here the method pays of a second time.
The most promising approach seems to be a mixture of both, but we could not establish a heuristic for the quantities involved.
Slow convergence of the semismooth Newton method can also be observed if the initial viscosity is chosen to be rather large ($\varepsilon_0 = 1$). Then the active and inactive sets change drastically during the outer loop, causing a large number of steps in the inner iteration. For certain examples we observed iteration numbers larger than 150.
Considering the multipliers of Example 6.4 and 6.5 in Figure 6.9 and 6.14, the irregular behavior of the multipliers at the parts of the boundary there the vector $\mathbf{b}$ points inwards the domain can be observed. This artifact clearly violates the complementarity condition since

$$(y, \lambda) \neq 0$$

holds (recall, that the plots only depict the values on the inner nodes). This is a direct consequence of the chosen regularization-penalization approach for the problem of optimal control. The relaxed product constraint $(y, \xi) \leq \alpha$ always provides a small threshold for ξ to be positive were y is positive as well. We have already pointed out that for low values of ε the underlying problem can be interpreted as transport of the material distribution $u + \xi + f$ in direction $\mathbf{b}$. Since y and u and thus ξ are forced to satisfy homogeneous boundary conditions at each step of the algorithm, the difference of data and state is largest near the boundary. To compensate this difference, the system tries to build up material $u + \xi + f$ near to the boundaries, where $\mathbf{b}$ points inwards the domain Ω. Here the threshold α is used and the suspicious shape of the multipliers is based on. We point out that this seems to be a general issue of the penalization-regularization approach for nonsymmetric differential operators since it can also be observed in [95, 73]. Moreover, the size of the feature strongly depends on the initial choice of α_0 but will be present for any positive value. We tested the algorithm for different α_0 and observed, that the height of the artifact decreased for decreasing α_0. Unfortunately, it is not possible to choose an update strategy for α such that the artifact vanishes from a certain discretization on. This is based on the coupling of the parameter to γ by Assumption 6.1. The feature would only disappear if α would decrease with higher order than assumed but then the convergence of stationary points according to Theorem 6.2.3 would no longer be guaranteed and thus no convergence of the algorithm would hold.

7 Optimal Control of non stationary Hyperbolic Variational Inequalities

Introduction

In this section we will investigate the existence of stationary points of an optimal control problem subject to a first order hyperbolic variational inequality of the first kind of evolutionary type. Let $\Omega \subset \mathbb{R}^n$, $n \leq 3$ be an open subset with Lipschitz boundary. The variational inequality problem will be defined by a closed and convex set of the form

$$\mathcal{K} = \{v \in \mathfrak{V} | v(t) \in \mathbf{K} \text{ for a.e. } t \in (0,T)\}.$$

Here, $\mathbf{K}$ represents a closed and convex subset of $H_0^1(\Omega)$ to be made precise later. Note, that $\mathbf{K}$ remains fixed for $t \in (0,T)$ since a variation of it is beyond the scope of this thesis and will not be considered. The problem of optimal control we are interested in is given as

$$\begin{array}{lrll} \inf & \frac{1}{2}|y-y^d|^2_{L^2(\mathcal{Q})} + \frac{\tilde{\beta}}{2}\|y\|^2_{L^2(0,T;H_0^1(\Omega))} + \frac{\beta}{2}|u|^2_{L^2(\mathcal{Q})} & = \mathcal{J}(y,u) & \\ \text{s.t.} & y \in \mathcal{K}, \quad \langle D_t y + \mathbf{b}\cdot\nabla y + b^0 y - f - u, v - y\rangle & \geq 0 & \forall v \in \mathcal{K} \\ & y(0) & = u_0 & \end{array} \tag{7.1}$$

For the derivation of stationarity conditions we will restrict ourselves to the obstacle problem and, as in the preceding chapter, it consists of two steps. First we will regularize the differential operator providing an underlying problem of parabolic type by adding a weighted Laplace operator with respect to the spatial variables. We will then study the behavior of stationary points for the resulting problems

$$\begin{array}{lrll} \inf & \mathcal{J}(y,u) & & \\ \text{s.t.} & y \in \mathcal{K}, \quad \langle D_t y - \varepsilon\Delta y + \mathbf{b}\cdot\nabla y + b^0 y - f - u, v - y\rangle & \geq 0 & \forall v \in \mathcal{K} \\ & y(0) & = u_0 & \end{array} \tag{7.2}$$

for the weight ε tending to zero. This method is known as method of vanishing viscosity. (7.2) will be handled with regularization and penalization methods as utilized for example in [73, 95]. We extend the corresponding theory slightly since the case of non-symmetric differential operators, given in our problem, has not been considered so far.

The organization of this chapter will be as follows. In Section 7.1 we fix the notation and discuss solution concepts for general parabolic variational inequalities. In Section 7.2 we define the variational inequalities under investigation. We will study the existence of solutions for the hyperbolic and the parabolic problems, provide stability results and establish a certain kind of consistency. Section 7.3 is dedicated to the problems of optimal control. Besides proving the existence of solutions to (7.1) and (7.2) for certain initial conditions u_0 we will derive a consistency result for these problems as well. Finally we will obtain a weak form of stationarity system for (7.1) in a constructive way in Section 7.4.

7.1 Preliminaries

Consider the open and bounded domain $\Omega \subset \mathbb{R}^n$ with $1 \leq n \leq 3$ and Lipschitz boundary $\partial\Omega$. Recall the definition of the function spaces

$$\mathfrak{V} = L^2(0,T;H_0^1(\Omega)) \subset L^2(\mathcal{Q}) = L^2(\mathcal{Q})^* \equiv \mathfrak{H}^* \subset L^2(0,T;H^{-1}(\Omega)) = \mathfrak{V}^*$$

for a given $T > 0$ with corresponding norms $\|\cdot\| = \|\cdot\|_{\mathfrak{V}}$ and $|\cdot| = \|\cdot\|_{\mathfrak{H}}$ and $W(0,T)$ from Chapter 2.1.4. A general variational inequality of the first kind of evolution type with a differential operator $\mathcal{A} : V \to V^*$ is given as

$$\text{find } y \in \mathfrak{V} \text{ with } \langle D_t y + \mathcal{A}y - f, v - y\rangle_{\mathfrak{V}^*,\mathfrak{V}} \geq 0 \text{ for all } v \in \mathcal{K} \quad y(0) = u_0 \tag{7.3}$$

where $\mathcal{K} \subset \mathfrak{V}$ denotes an arbitrary closed and convex set. Initially, since $\mathcal{K} \subset \mathfrak{V}$, we are looking for solutions $y \in \mathfrak{V}$ of (7.3). Thus the existence of distributional derivatives with respect to t is not ensured at all. The following definition addresses this difficulty.

Definition 7.1.1. *A function $y \in L^2(0,T;V), y \in \mathcal{K}$ is called **weak solution** of* (7.3) *if*

$$\langle D_t v + \mathcal{A}y - f, v - y\rangle_{\mathfrak{V}^*,\mathfrak{V}} \geq 0$$

holds for all $v \in \mathcal{K} \cap W(0,T)$ with $v(0) = u_0$.

As usual for weak formulations, the regularity requirements on the primal objects are lowered and transferred to test functions. If a higher regularity of y with respect to t is ensured, the formulation can be strengthened.

Definition 7.1.2. *A function $y \in W(0,T), y \in \mathcal{K}$ is called **solution** of* (7.3) *if*

$$\langle D_t y + \mathcal{A}y - f, v - y\rangle_{\mathfrak{V}^*,\mathfrak{V}} \geq 0 \qquad y(0) = u_0$$

holds for all $v \in \mathcal{K}$.

Both types of solutions are related according to the following result.

Lemma 7.1.1. *Any solution in the sense of 7.1.2 is a solution in the sense of 7.1.1.*

The proof can be found for example in [95, 120] and will be omitted. If the operator $\mathcal{A}$ is elliptic, the existence of weak solutions to (7.3) is ensured under mild conditions. The corresponding theory can, for example, be found in [58].

Theorem 7.1.1. *Let $\mathcal{A} : V \to V^*$ be an elliptic, linear operator satisfying $(\mathcal{A}v, v) \geq \varepsilon\|v\|_V^2$ with $\varepsilon > 0$.*
For every $f \in \mathfrak{H}$, $u_0 \in \mathbf{K} \cap V$ there exist a unique weak solution to (7.3) *in the sense of Definition 7.1.1.*

After discussing solutions concepts for parabolic variational inequalities we present a result which allows for switching between certain formulations of the parabolic objects. This will be used at several points in the remainder of the chapter without being noted explicitly.

Lemma 7.1.2. *The solutions (Definition 7.1.2) to the following problems are equivalent. Find* $y \in W(0,T)$

$$y(t) \in \mathbf{K}, \langle D_t y(t) + \mathcal{A}y(t) - f(t), v - y(t)\rangle \geq 0 \textit{ f.a. } v \in \mathbf{K} \textit{ f.a.e. } t \in [0,T], \quad y(0) = u_0 \tag{7.4}$$

and find $y \in W(0,T) \cap \mathcal{K}$ *with*

$$\int_0^T \langle D_t y + \mathcal{A}y - f, v - y\rangle \geq 0 \textit{ f.a. } v \in \mathcal{K}, \quad y(0) = u_0 \tag{7.5}$$

Proof. The proof is similar to [25, Appendix 1]. The implication (7.4)⇒(7.5) is obvious. For the converse, we consider an arbitrary Lebesgue point $\hat{t} \in (0,T)$. Since the time interval is open, we find subsets $\Theta \subset (0,T)$ with $\hat{t} \in \Theta$. Moreover, we choose $\tilde{v} \in \mathbf{K}$ arbitrary. For any suitable Θ, we define $v(t) = \eta(t)\tilde{v}$ with $\eta \in C_c^\infty(\Theta)$ and $\eta(\hat{t}) = 1, \|\eta\|_\infty = 1$ which provides $v(t) \in \mathcal{K}$. The same holds true for the test function

$$w(t) = \begin{cases} v(t) & t \in \Theta \\ y(t) & t \in (0,T)\backslash\Theta \end{cases}$$

Substituting into (7.5) we obtain

$$\int_\Theta \langle D_t y + \mathcal{A}y - f, v - y\rangle \geq 0. \tag{7.6}$$

By the Lebesgue differentiation Theorem provides, for any Lebesgue point

$$\lim_{|\Theta|\to 0} \frac{1}{|\Theta|} \int_\Theta \phi(s)ds = \phi(\hat{t}).$$

holds if $\phi \in L^1(0,T)$ and $\Theta = B(x;r)$. Defining

$$\phi(s) = \langle D_t y(s) + \mathcal{A}y(s) - f(s), v(s) - y(s)\rangle$$

(7.6) clearly implies $\phi(\hat{t}) \geq 0$. Since almost every point $t \in (0,T)$ is a Lebesgue point and $\tilde{v} \in \mathbf{K}$ was chosen arbitrarily, the asserted equivalence is proven. □

At this point we discuss the requirements on objective functional $\mathcal{J}$ for the optimization problems. Similar to the stationary case, regularity of the state y_ε for decreasing viscosity parameters ε can only be maintained, if a certain boundedness of the state is ensured by the optimization problem. Considering (7.2) we note, that in contrast to the stationary case $(\tilde{P}^\varepsilon)$ we do not have to bound the states y_ε in $W(0,T)$, the usual space of solutions for the underlying parabolic variational inequality problems, but merely in $\mathfrak{V}$. The sufficiency of this bound will be carried out in detail in the upcoming sections. Consequently, any Fréchet differentiable functional implying a bound on the state of the underlying system in $\mathfrak{V}$ is sufficient for the following procedure. For reasons of readability we focused on a tracking type functional with a Tikhonov term for y in $\mathfrak{V}$ as discussed in the preceding chapter for the time-independent case.

7.2 The Variational Inequalities

In this section we will analyze the auxiliary parabolic variational inequalities (7.3) with respect to existence of unique solutions for three different forms of the set $\mathbf{K}$ given as

$$\begin{aligned}\mathbf{K}_1 &= \{v \in V | v(x) \geq \psi(x) \text{ for a.e. } x \in \Omega\},\\ \mathbf{K}_2 &= \{v \in V || \nabla v(x)| \leq c \text{ for a.e. } x \in \Omega\},\\ \mathbf{K}_3 &= \mathbf{K}_1 \cap \mathbf{K}_2.\end{aligned}$$

For $\mathbf{K}_1$, the resulting variational inequality problem corresponds to an obstacle problem, where the function $\psi \in W^{2,p}(\Omega)$ with $p > n$ represents the obstacle. To ensure the set $\mathbf{K}_1$ to be non empty, we in addition assume $\psi(x) \leq 0$ for all $x \in \partial\Omega$. Besides this problem, which is the main subject of the entire section concerning optimal control, we show, that the corresponding theory also is applicable to constant gradient constraints ($\mathbf{K}_2$) and a combination of both ($\mathbf{K}_3$). A useful interpretation of $\mathbf{K}_2$ in the context of first order hyperbolic differential operators is piling of material which only can have a certain slope (see Section 4.2 for details). Finally, the combination $\mathbf{K}_3$ can seen as the modeling of sand rippling into a hourglass where, besides the gradient constraint introduced by the stiction of the sand, the obstacle in form of the hourglass influences the process.
Usually variational inequalities are defined by data $\tilde{f}$ of a certain regularity. We will always split this term into some fixed part $f \in L^2(Q)$ and some part u which can be influenced. Given $u, f \in L^2(\mathcal{Q})$, we search for a solution $y \in W(0,T) \cap \mathbf{K}$ of the hyperbolic variational inequality

$$\begin{aligned}\langle D_t y(t) + \mathbf{b} \cdot \nabla y(t) + b_0 y(t) - f(t) - u(t), v - y(t)\rangle &\quad 0 \qquad \forall v \in \mathbf{K} \text{ a.e. } t \in [0,T]\\ y(0) &= u_0\end{aligned} \tag{VI^T}$$

To obtain this function, we will approximate the problem by the following family of parabolic variational inequalities. Find $y_\varepsilon \in W(0,T) \cap \mathbf{K}$ satisfying

$$\begin{aligned}\langle D_t y_\varepsilon(t) - \varepsilon\Delta y_\varepsilon(t) + \mathbf{b} \cdot \nabla y_\varepsilon(t) + b_0 y_\varepsilon(t) - f(t) - u(t), v - y_\varepsilon(t)\rangle &\geq 0\\ y(0) &= u_0\end{aligned} \tag{VI_ε^T}$$

for all $v \in \mathbf{K}$ and almost every $t \in (0,t)$.

7.2.1 The Approximating Parabolic Variational Inequalities

In this part we proof the existence of solutions to (VI_ε^T). For the remainder we will abbreviate the differential operator with respect to the spatial variables in the following way.

$$A^\varepsilon y = -\varepsilon\Delta y + \mathbf{b} \cdot \nabla y + b^0 y = -\varepsilon\Delta y + A^0 y$$

We will use the fact, that the image of A^0 for arguments $y \in \mathfrak{V}$ is in $L^2(\mathcal{Q})$. The vector field $\mathbf{b}$ is assumed to have the same regularity properties as in Chapter 6. A^ε is a non symmetric differential operator and we have to restrict our self to a certain setting in order to obtain the existence of solutions of the underling problem. We recall the definition of the coercivity condition from the preceding chapter suited to the parameter dependent case.

Definition 7.2.1. *The differential operator fulfills the* ***coercivity condition*** *if*

$$\sum_{i=1}^{n} b^i_{x_i}(x,t) \leq 2b^0(x,t)$$

is satisfied almost everywhere in $\mathcal{Q}$. If the inequality is fulfilled strictly, i.e. there exist some $\underline{b}$ such that

$$2b^0(x,t) - b^i_{x_i}(x,t) \geq \underline{b} > 0$$

holds for every $(x,t) \in Q$, the operator fulfills the ***strong coercivity condition****.*

Remark 7.2.1. *The quadratic form introduced by the nonsymmetric differential operator A^ε,*

$$Q^\varepsilon : \mathfrak{V} \to \mathbb{R}, \qquad Q^\varepsilon(v) = \langle A^\varepsilon v, v\rangle_{\mathfrak{V}^*,\mathfrak{V}}$$

is convex provided A^ε satisfies the coercivity condition. This is remarkable since in general, quadratic forms introduced by non symmetric operators do not have this property. $L^p(\Omega)$ with $p > 1$ for example only ensures, that any continuous quadratic form is delta-convex (i.e. the difference of two convex functions) which has been shown in [91]. In the case at hand, the convexity follows from

$$\int_0^T \langle A^\varepsilon v(t), v(t)\rangle dt = \int_0^T \langle -\varepsilon\Delta v(t) + [b^0 - (1/2)\nabla\cdot\mathbf{b}]v(t), v(t)\rangle dt$$

i.e. in the quadratic case the nonsymmetric perturbation of the operator $-\varepsilon\Delta y + b^0 y$ can be absorbed in the zero order term and the form Q^ε is equivalent to one introduced by a symmetric operator. Convexity is now a consequence of the Theorem of Kachurovskii (see, e.g. [144]).

Until now it seems as standard existence theory as [7] can be applied. Unfortunately, this theory is always suited to symmetric operators $\mathcal{A}$ and we have to argue in a different way for several reasons. The adjoint operator $(A^\varepsilon)^*$ is given as

$$(A^\varepsilon)^* p = -\varepsilon\Delta p - \mathbf{b}\cdot\nabla p + (b_0 - \nabla\cdot\mathbf{b})p.$$

As A^ε is maximal monotone (see Lemma 6.1.1 and comments below), so is $(A^\varepsilon)^*$. First we establish a result proving, that A^ε is well related to the feasible set. The proof can be found in [27].

Lemma 7.2.1. *Let $v \in \mathbf{K}_i$, $i \in \{1,2,3\}$ denote an arbitrary element from one of the feasible sets.*
Then there exists an $\bar{\varepsilon} > 0$ such that for any $\lambda > 0$ and $\varepsilon \in (0, \bar{\varepsilon}]$ the solution $\tilde{v}_\lambda$ of

$$(I + \lambda A^\varepsilon)\tilde{v}_\lambda = v$$

satisfies $\tilde{v} \in \mathbf{K}_i$.

For $\mathbf{K}_2$ and consequently $\mathbf{K}_3$ we have to assume a higher regularity of the boundary of the domain. The result only applies for $\partial\Omega \in C^2$.
A first existence result concerning general parabolic variational inequalities of the type (7.3) is stated below and can be found in [25].

Theorem 7.2.1 (Theorem II.9)**.** *Let $\mathcal{A} : H_0^1(\Omega) \to H^{-1}(\Omega)$ be a monotone and hemicontinuous operator (in the sense of Definition 2.3.1) satisfying*

$$\begin{aligned} \langle \mathcal{A}v, v\rangle &\geq \alpha\|v\|^2 - C \text{ for all } v \in H_0^1(\Omega) \\ \|\mathcal{A}v\|_{H^{-1}(\Omega)} &\leq C(\|v\|^{p-1} + 1) \text{ with } p \geq 2 \\ (I + \lambda\mathcal{A})^{-1}\mathbf{K} &\subset \mathbf{K} \end{aligned}$$

Then for any $f \in \mathfrak{H}$ and $u_0 \in \overline{\mathbf{K}}^{L^2(\Omega)}$ there exist a unique $y \in W(0,T)$ satisfying

$$\langle D_t y + \mathcal{A}y, v - y\rangle \geq \langle f, v - y\rangle \text{ for a.e. } t \in (0,T), \forall v \in \mathbf{K}, \quad y(0) = u_0.$$

In addition, we have

$$D_t y + \mathcal{A}y \in \mathfrak{H} \tag{7.7}$$

$$|D_t y(t) + \mathcal{A}y(t)| \leq |f(t)| \text{ for a.e. } t \in (0,T). \tag{7.8}$$

Remark 7.2.2. *Due to Lemma 6.1.1 and Lemma 7.2.1 the result can be applied directly to (VI_ε^T) for the sets $\mathcal{K}$ defined by $\mathbf{K}_1, \mathbf{K}_2, \mathbf{K}_3$.*

Unfortunately, this result does not provide, that the solution y is an element of the feasible set $\mathbf{K}$ for almost every time $t \in (0,T)$. Moreover, we do not obtain the regularity of $D_t y \in L^2(\mathcal{Q})$ and $\mathcal{A}y \in L^2(\mathcal{Q})$ alone as the quantities might share a singularity such that the sum of both is in $L^2(\mathcal{Q})$ while the terms alone fail to possess this regularity. By the following existence result from [7, 25] for symmetric differential operators we will overcome this issue.

Theorem 7.2.2 (Theorem II.8 & Corollary II.2)**.** *Let $\mathcal{A} : V \to V^*$ be a symmetric, linear and monotone operator satisfying*

$$\langle \mathcal{A}v, v\rangle \geq \varepsilon\|v\|^2.$$

Let $\phi_0 : H_0^1(\Omega) \to (-\infty, \infty]$ be a convex lower semicontinuous function, $\phi_0 \not\equiv \infty$. Then for any $f \in \mathfrak{H}$ and $u_0 \in K \cap V$ there exist a unique function $y \in W(0,T)$ satisfying

$$\langle D_t y + A^\varepsilon y, v - y\rangle + \phi_0(v) - \phi_0(y) \geq \langle f, v - y\rangle \text{ for a.e. } t \in (0,T), \forall v \in \mathbf{K}, \ \ y(0) = u_0.$$

In $D_t y \in \mathfrak{H}$ satisfies

$$\left(\int_0^T |D_t y|^2\right)^{1/2} \leq \left(\int_0^T |f|^2\right)^{1/2} + ((1/2)(\mathcal{A}u_0, u_0))^{1/2} \tag{7.9}$$

After providing both results, we present theory ensuring the existence of solutions to (VI_ε^T) satisfying $y \in \mathcal{K}$.

Proposition 7.2.1. *Let the assumptions on A^ε from Theorem 7.2.1 and 7.2.2 be satisfied. For any $f \in L^2(\mathcal{Q})$ there exits a unique $y \in W(0,T) \cap \mathcal{K}$ satisfying*

$$\langle D_t y + A^\varepsilon y - f, v - y\rangle_{\mathfrak{Y}^*,\mathfrak{Y}} \geq 0$$

Proof. Consider an arbitrary $f \in \mathfrak{H}$ and $u_0 \in H_0^1(\Omega) \cap \mathbf{K}$. Due to Remark 7.2.2 there exists a unique $y \in W(0,T)$ satisfying

$$\langle D_t y + A^\varepsilon y, v - y\rangle \geq \langle f, v-y\rangle \text{ for a.e. } t \in (0,T), \forall v \in \mathbf{K}, \quad y(0) = u_0.$$

Since $y \in L^2(0,T;H_0^1(\Omega))$ and $\mathbf{b} \in (L^\infty(\Omega))^n$, $\mathbf{b}\cdot\nabla y(t) \in L^2(\Omega)$ is satisfied for almost every $t \in (0,T)$. Consequently y also solves

$$\langle D_t y - \varepsilon\Delta y + b^0 y, v - y\rangle \geq \langle (f - \mathbf{b}\cdot\nabla y), v-y\rangle \text{ for a.e. } t \in (0,T), \forall v \in \mathbf{K}, \quad y(0) = u_0.$$

Now consider $\tilde{f} = f - \mathbf{b}\cdot\nabla y \in \mathfrak{H}$ and $\tilde{A}^\varepsilon = -\varepsilon\Delta\cdot + b^0\cdot$. The operator satisfies the same estimates concerning monotonicity and boundedness as A^ε. Moreover, the inclusion property from Lemma 7.2.1 holds as well. An application of Theorem 7.2.1 yields the existence of an unique $\tilde{y} \in W(0,T)$ with

$$\langle \tilde{D}_t y + \tilde{A}^\varepsilon \tilde{y}, v - \tilde{y}\rangle \geq \langle \tilde{f}, v - \tilde{y}\rangle \text{ for a.e. } t \in (0,T), \forall v \in \mathbf{K}, \quad y(0) = u_0.$$

Thus $\tilde{y}$ and y have to coincide. Finally, applying Theorem 7.2.2 to $\tilde{f}$ and $\tilde{A}^\varepsilon$ we obtain a unique solution $\hat{y} \in W(0,T)$ satisfying

$$\langle \hat{D}_t y + \tilde{A}^\varepsilon \hat{y}, v - \hat{y}\rangle + \phi_0(v) - \phi_0(\hat{y}) \geq \langle \tilde{f}, v - \hat{y}\rangle \text{ for a.e. } t \in (0,T), \forall v \in \mathbf{K}, \ \ y(0) = u_0.$$

For the indicator function of the closed convex set $\phi_0 = I_{\mathbf{K}}$, the latter problem is equivalent to (7.4) and thus in the sense of (7.5). Since all of the solutions are unique, we find

$$y = \tilde{y} = \hat{y}$$

implying, that the initial solution has to satisfy $y \in \mathcal{K}$. □

The essential observation in the preceding proof is, that the regularity of y with respect to the spatial variables and of the vector field $\mathbf{b}$ allows to consider $\mathbf{b}\cdot\nabla y$ as an element of $L^2(\mathcal{Q})$.
Note that the existence of solutions to the variational inequalities can be also established by referring to [144, chapter III.7] where the existence of weak solutions in the sense of Definition 7.1.1. was supplemented by a regularity result. Now by uniqueness of the weak solution the existence of a strong solution was established. However, Proposition 7.2.1 provides the estimates (7.8) and (7.10) which are essential for the following considerations and not covered by the given reference.
Next we present a fundamental result which will be useful even for the limit considerations concerning the viscosity parameter. The Proposition establishes the boundedness of $D_t y$ in $L^2(\mathcal{Q})$ for solutions of (VI_ε^T) under certain conditions.

Lemma 7.2.2. *If the solution y of (VI_ε^T) is bounded in $\mathfrak{V}$ by c and the corresponding data (u_0, f) are elements of $H_0^1(\Omega) \times L^2(\Omega)$, the generalized time derivative of y is bounded in $L^2(\mathcal{Q})$ independently of ε.*

Proof. Since any y solves (VI_ε^T), it satisfies in particular

$$\begin{aligned} \langle D_t y(t) - \varepsilon\Delta y(t) + b^0 y(t) + \tilde{f}, v - y(t)\rangle &\geq 0 \text{ for all } v \in K, a.e. t \in (0,T) \\ y(0) &= u_0 \end{aligned}$$

for $\tilde{f} = \mathbf{b} \cdot \nabla y(t) - f \in L^2(\mathcal{Q})$. The differential operator $\tilde{A}^\varepsilon \cdot = -\varepsilon\Delta \cdot + b^0 \cdot$ artificially acting on $y(t)$ fulfills the requirements of Theorem 7.2.2 and the unique solution $\tilde{y}$ to

$$\langle \tilde{D}_t y + \tilde{A}^\varepsilon \tilde{y}, v - \tilde{y} \rangle \geq \langle f, v - \tilde{y} \rangle \text{ for a.e. } t \in (0,T), \forall v \in K, \quad \tilde{y}(0) = u_0.$$

has to be equal to y. Utilizing (7.9) we find

$$\left(\int_0^T |D_t y|^2 \right)^{1/2} \leq \left(\int_0^T |\tilde{f}|^2 \right)^{1/2} + ((1/2)(A^\varepsilon u_0, u_0))^{1/2}. \tag{7.10}$$

Using the assumed bound on y we can now estimate the right hand side

$$\begin{aligned} \leq & c + \left(|f|^2 + \int_0^T \int_\Omega (\mathbf{b} \cdot \nabla y(t))^2 dx dt - 2 \int_0^T \int_\Omega \mathbf{b} \cdot \nabla y(t) f(t) dx dt |f(t)| \right)^{1/2} \\ \leq & c + \left(|f|^2 + |\mathcal{Q}| \overline{|\mathbf{b}|}^2 c^2 + C \overline{|\mathbf{b}|} c \| f(t) \| \right)^{1/2} \leq \tilde{c} \end{aligned}$$

where c only depends on $|f|$ and $\|u_0\|$ but not on ε. □

A similar estimate as (7.10) can be found for example in [137].

Remark 7.2.3. *For fixed ε we combine* (7.7) *and Lemma 7.2.2 to obtain*

$$A^\varepsilon y \in L^2(\mathcal{Q})$$

for any solution of (7.3) *provided the initial condition is regular enough and y bounded in $\mathfrak{V}$.*

7.2.2 Evolutionary Variational Inequalities with First Order Operators

As in Chapter 6 we introduce a certain class of regular solutions to the hyperbolic problem.

Definition 7.2.2. *Consider the variational inequality*

$$y \in \mathcal{K}, \langle D_t y + A^0 y - f, v - y \rangle \geq 0 \textit{ for all } v \in \mathcal{K}, y(0) = u_0$$

for a closed convex set $\mathcal{K} \subset \mathfrak{V}$ and given data $f \in L^2(\mathcal{Q}), u_0 \in H_0^1(\Omega)$.
$y \in W(0,T)$ is called ***solution in the viscosity sense*** *if there exist a sequence of viscosity parameters $\varepsilon \to 0$ and pairings $(y_\varepsilon, f_\varepsilon) \in W(0,T) \times L^2(\mathcal{Q})$ satisfying the parabolic variational inequalities*

$$y_\varepsilon \in \mathcal{K}, \langle D_t y_\varepsilon + A^\varepsilon y_\varepsilon - f_\varepsilon, v - y_\varepsilon \rangle \geq 0 \textit{ for all } v \in \mathcal{K}, y(0) = u_0$$

and having the following convergence properties.

$$y_\varepsilon \rightharpoonup y \textit{ in } W(0,T) \textit{ and } f_\varepsilon \rightharpoonup f \textit{ in } L^2(\mathcal{Q})$$

Now we establish a result concerning uniqueness of solutions to the underlying hyperbolic variational inequality.

Lemma 7.2.3. *Let $f \in \mathfrak{H}$ and $u \in \mathfrak{H}$ be given.*
Under the strong coercivity condition any solution of (VI^T) is unique in $W(0,T)$.

Proof. The proof follows classical arguments. Let y^1 and y^2 denote two solutions of the (VI^T). Since $y^1, y^2 \in \mathcal{K}$, we test the variational inequalities fulfilled by y^i with the other solution and add up the results. Setting $v = y^1 - y^2$ we obtain

$$0 \geq \langle D_t v + \mathbf{b} \cdot \nabla v + b^0 v, v\rangle = \frac{1}{2}\|v(T)\|^2_{L^2(\Omega)} + \langle (b^0 - \frac{1}{2}\sum_{i=1}^{n} b^i_{x_i})v, v\rangle \geq \frac{1}{2}\|v(T)\|^2_{L^2(\Omega)} + \underline{b}\langle v, v\rangle$$

for $\underline{b} = \inf_{x\in\Omega}\{b^0 - (\sum\limits_{i=1}^{n} b^i_{x_i}/2)\}$ which is positive under strict coercivity. Consequently we have $\|v\|^2 = 0$ ensuring $y^1 = y^2$ in $\mathfrak{V}$ which implies $y^1(x) = y^2(x)$ almost everywhere. Thus by Definition 2.1.1 the generalized time derivatives satisfy

$$\int_0^T \varphi(s)\langle D_t y^1(s), v\rangle = (-1)\int_0^T \varphi_t(y^1(s), v) = (-1)\int_0^T (y^2(s), v) = \int_0^T \varphi(s)\langle D_t y^2(s), v\rangle$$

for all $\varphi \in C_c^\infty(0,T)$ and $v \in \mathfrak{V}$ demonstrating their equivalence in $\mathfrak{V}^*$. Consequently, y^1 and y^2 coincide in $W(0,T)$. □

As a consequence of the preceding result, the terminal state $y(T)$ is unique as long as the coercivity condition is ensured to hold. The following part establishes convergence properties of feasible pairings

$$(y_\varepsilon, u_\varepsilon) \in W(0,T) \times \mathfrak{H}$$

for a decreasing sequence of viscosity parameters ε under suitable conditions.

Lemma 7.2.4. *Consider an arbitrary bounded sequence of nonegative viscosity parameters $\{\varepsilon\}$ and corresponding pairings $\{(y_\varepsilon, u_\varepsilon)\}$ satisfying (VI^T_ε) for fixed data f, u_0.*
If $\{y_\varepsilon\}$ and $\{u_\varepsilon\}$ are bounded independent of ε in $\mathfrak{V}$ and $L^2(\mathcal{Q})$ respectively, the sequence $\{D_t y_\varepsilon\}$ is bounded in $L^2(\mathcal{Q})$ independent of ε.

Proof. This result is a direct consequence on Lemma 7.2.2 since the assumptions yield, that the right hand side of (7.10) can be estimated independently of $\varepsilon < \infty$. □

Before proving the main result of this section, we show, that a sequence of feasible pairings to (VI^T_ε) for a decreasing sequence of viscosity parameters converges to a solution of the hyperbolic variational inequality under certain assumptions concerning their boundedness.

Lemma 7.2.5. *Consider a sequence of nonegative viscosity parameters $\varepsilon \to 0$ and pairings $\{y_\varepsilon, u_\varepsilon\}$ satisfying (VI^T_ε). Let the sequence $\{y_\varepsilon, u_\varepsilon\}$ be bounded in $\mathfrak{V} \times \mathfrak{H}$.*
Then there exist weak accumulation points $(\tilde{y}, \tilde{u}) \in W(0,T) \times \mathfrak{H}$ satisfying

$$\langle D_t\tilde{y}(t) + \mathbf{b}\cdot\nabla\tilde{y}(t) + b^0\tilde{y}(t) - f(t) - \tilde{u}(t), v - \tilde{y}(t)\rangle \geq 0 \; for\ all\ v \in \mathbf{K}, a.e. t \in (0,T)$$
$$y(0) = u_0$$

Proof. By Proposition 7.2.1, for any $\varepsilon > 0$ the solution y_ε of (VI^T_ε) is an element of $W(0,T)$. According to Lemma 7.2.4, $\{D_t y_\varepsilon\}$ is bounded in $\mathfrak{H} \subset \mathfrak{V}^*$. Consequently, y_ε is bounded in $W(0,T)$ and there exist weakly converging subsequence again denoted by ε and weak accumulation points $(\tilde{y},\tilde{u}) \in W(0,T) \times \mathfrak{H}$ with $y_\varepsilon \rightharpoonup \tilde{y}$ in $W(0,T)$ and $u_\varepsilon \rightharpoonup \tilde{u}$ in $\mathfrak{H}$. Moreover, since we have a bound on $D_t y_\varepsilon$ in $\mathfrak{H}$, the weak convergence

$$D_t y_\varepsilon \rightharpoonup D_t \tilde{y} \text{ in } \mathfrak{H}$$

holds by Proposition 2.1.2. Considering (VI_ε^T) for an arbitrary $v \in \mathfrak{V} \cap \mathcal{K}$ we find

$$\langle D_t y_\varepsilon + A^\varepsilon y_\varepsilon - f - u_\varepsilon, v - y_\varepsilon \rangle \geq 0$$

Rearranging and considering the corresponding duality pairings provides

$$(D_t y_\varepsilon, v - y_\varepsilon) + (\mathbf{b} \cdot \nabla y_\varepsilon + b^0 y_\varepsilon, v - y_\varepsilon) - (f, v - y_\varepsilon) - (u_\varepsilon, v - y_\varepsilon) \geq \varepsilon \langle \Delta y_\varepsilon, v \rangle - \varepsilon \langle \Delta y_\varepsilon, y_\varepsilon \rangle.$$

The compact embedding $W(0,T) \to \mathfrak{H}$ (see Lemma A.2.1) yields the existence of a further subsequence $\{\varepsilon\}$ such that $y_\varepsilon \to \tilde{y}$ strongly in $\mathfrak{H}$ along this subsequence. Now $(\mathbf{b} \cdot \nabla y_\varepsilon + b^0 y_\varepsilon, v - y_\varepsilon) = (-\mathbf{b} \cdot \nabla v + (b^0 - \nabla \cdot \mathbf{b})v, y_\varepsilon) - ((b^0 - (1/2)\nabla \cdot \mathbf{b})y_\varepsilon, y_\varepsilon)$, the strong convergence $y_\varepsilon \to \tilde{y}$ in $\mathfrak{H}$ and $\varepsilon\langle \Delta y_\varepsilon, v\rangle - \varepsilon \langle \Delta y_\varepsilon, y_\varepsilon \rangle \geq -\varepsilon \|y_\varepsilon\| \|v\|$ as well as the Hölder inequality provide

$$\langle D_t \tilde{y} + A^0 \tilde{y} - f - \tilde{u}, v - \tilde{y} \rangle \geq 0$$

since $\|y_\varepsilon\|$ is bounded by assumption.
To prove, that the initial condition is satisfied by $\tilde{y}$ we consider an arbitrary $\varphi \in C_c^\infty(\Omega)$ and a smooth function $\eta \in C^\infty(0,T)$ with $\eta(0) = 1$ and $\eta(T) = 0$. $w(x,t) = \eta(t)\varphi(x)$ is an element of $W(0,T)$ and testing (VI^T), (VI_ε^T) with it we obtain after subtraction

$$\begin{aligned} 0 &= \lim_{\varepsilon \to 0} (D_t(y_\varepsilon - \tilde{y}) - \varepsilon \Delta y_\varepsilon + A^0(y_\varepsilon - \tilde{y}) - (u_n - u), w) \\ &= \underbrace{\lim_{\varepsilon \to 0} (-\varepsilon \Delta y_\varepsilon + A^0(y_\varepsilon - \tilde{y}) - (u_n - u), w)}_{=0} - \underbrace{\lim_{\varepsilon \to 0} \langle D_t w, y_\varepsilon - \tilde{y} \rangle}_{=0} + \lim_{\varepsilon \to 0} (u_0 - \tilde{y}(0), \varphi). \end{aligned}$$

Thus $(u_0 - \tilde{y}(0), \varphi)_{L^2(\Omega)} = 0$ for all $v \in C_c^\infty(\Omega)$. The variational Lemma (Theorem A.4.12) now yields $\tilde{y}(0) = u_0$ almost everywhere in Ω. □

Next we show, similar to the preceding chapter, that any paring $(y,u) \in W(0,T) \times L^2(\mathcal{Q})$ satisfying (VI^T) for fixed $f \in L^2(\mathcal{Q})$ represents a solution in the viscosity sense for given data $f + u$. Again we use a modification of results from [133] concerning degenerate differential operators for time independent problems.

Theorem 7.2.3. *Let A^0 satisfy the strong coercivity condition and let $(y,u) \in W(0,T)$ satisfy the variational inequality*

$$y \in \mathcal{K}, \langle D_t y + A^0 y - f - u, v - y \rangle \geq 0 \textit{ for all } v \in \mathcal{K}, y(0) = u_0.$$

Moreover let $h \in L^2(\mathcal{Q})$ denote an arbitrary shift and consider a sequence $\varepsilon \to 0$.
Then $\{(y_\varepsilon, u_\varepsilon)\} \in W(0,T) \times L^2(\Omega)$ with $u_\varepsilon = u + \varepsilon h$ and $y_\varepsilon \in W(0,T)$ denoting the unique solution for these data in (VI_ε^T) satisfy the following convergence properties along a subsequence denoted the same.

$$y_\varepsilon \to y \textit{ in } \mathfrak{V}, \; y_\varepsilon \rightharpoonup y \textit{ in } W(0,T) \textit{ and } u_\varepsilon \to u \textit{ in } L^2(\mathcal{Q}).$$

Proof. The existence of solutions $y_\varepsilon \in W(0,T)$ is ensured by Proposition 7.2.1. The assumed regularity of y allows to test any (VI_ε^T) with it and (VI^T) can be tested with any y_ε respectively. Adding both we find by linearity of the differential operator

$$\langle D_t(y_\varepsilon - y) - \varepsilon\Delta y_\varepsilon + \mathbf{b}\cdot\nabla(y_\varepsilon - y) + b^0(y_\varepsilon - y) - \varepsilon h, y - y_\varepsilon\rangle \geq 0.$$

Adding and subtracting $\varepsilon\Delta y$ and rearranging we further get for $v_\varepsilon = y_\varepsilon - y$

$$\begin{aligned}\langle D_t v_\varepsilon - \varepsilon\Delta v_\varepsilon + \mathbf{b}\cdot\nabla v_\varepsilon + b^0 v_\varepsilon, v_\varepsilon\rangle &\leq \varepsilon\langle\Delta y, v_\varepsilon\rangle + \varepsilon(h, v_\varepsilon)\\ &\leq \varepsilon\|\Delta y\|_{\mathfrak{V}\to\mathfrak{V}^*}\|v_\varepsilon\| + \varepsilon|h|\|v_\varepsilon\|\\ &\leq \varepsilon/2(\|y\| + |h|)^2 + \varepsilon/2\|v_\varepsilon\|^2\end{aligned}$$

where the estimate for the operator norm follows from Theorem A.4.7. The last line is an application of Young's inequality. Due to Lemma 2.1.3 and the strong coercivity condition of A^0 we find

$$\frac{1}{2}(|y_\varepsilon(T) - y(T)|^2 - |y_\varepsilon(0) - y(0)|^2) + \varepsilon\|v_\varepsilon\|^2 + \underline{b}|v_\varepsilon|^2 \leq \langle D_t v_\varepsilon - \varepsilon\Delta v_\varepsilon + \mathbf{b}\cdot\nabla v_\varepsilon + b^0 v_\varepsilon, v_\varepsilon\rangle$$

providing

$$\begin{aligned}(1/2)|y_\varepsilon(T) - y(T)|^2 &\leq \varepsilon/2(\|y\| + |h|)^2\\ \underline{b}|v_\varepsilon|^2 &\leq \varepsilon/2(\|y\| + |h|)^2\\ \|v_\varepsilon\|^2 &\leq (\|y\| + |h|)^2\end{aligned}\tag{7.11}$$

The last inequality implies boundedness of v_ε in $\mathfrak{V}$ and the existence of some weak limit $\tilde{v} \in \mathfrak{V}$ such that

$$v_\varepsilon \rightharpoonup \tilde{v}$$

along a further subsequence denoted the same. This establishes a bound on y_ε in $\mathfrak{V}$ by the inverse triangle equation. The second inequality yields the strong convergence $y_\varepsilon \to y$ in $L^2(\mathcal{Q})$.

Since we can not rely on density results as in Theorem 6.1.2, in the setting of Lebesgue spaces of vector valued functions we have use a different argument at this point. Since $(y_\varepsilon, u_\varepsilon)$ is a sequence in $\mathfrak{V} \times L^2(\mathcal{Q})$ bounded independent of ε and $y_\varepsilon \in W(0,T)$ holds for any ε, the application of Lemma 7.2.5 provides the existence of a subsequence of viscosity parameters again denoted by ε and two elements $(\tilde{y}, \tilde{u})$ such that $y_\varepsilon \rightharpoonup \tilde{y}$ in $W(0,T)$ and $u_\varepsilon \rightharpoonup \tilde{u}$ in $L^2(\mathcal{Q})$ satisfying in the limit

$$\tilde{y} \in \mathcal{K}, \langle D_t\tilde{y} + A^0\tilde{y} - f - \tilde{u}, v - \tilde{y}\rangle \geq 0 \text{ for all } v \in \mathcal{K}, \tilde{y}(0) = u_0.$$

Since u_ε already is a strongly converging sequence, $\tilde{u} = u$ has to hold. By Lemma 7.2.3 the solution to (VI^T) is unique implying $\tilde{y} = y$ and thus providing

$$y_\varepsilon \rightharpoonup y \Leftrightarrow v_\varepsilon \rightharpoonup 0 \text{ in } \mathfrak{V}.$$

Now we obtain analogously to Theorem 6.1.2

$$\|v_\varepsilon\|^2 \leq \langle\Delta y, v_\varepsilon\rangle + (h, v_\varepsilon)$$

yielding the claimed strong convergence in $\mathfrak{V}$. □

Note that this result in addition establishes the strong convergence $\tilde{y}_\varepsilon(T) \to y(T)$ in $L^2(\Omega)$. As in the previous chapter, Theorem 7.2.3 essentially depends on the strong coercivity condition for the same reasons.

Remark 7.2.4. *The preceding result shows, that any solution in the viscosity sense $y \in W(0,T)$ for data f and control u, both in $\mathfrak{H}$, has a generalized time derivative in $L^2(\mathcal{Q})$. Considering Theorem 7.2.3 for $h \equiv 0$ we obtain a sequence of approximating solutions $y_\varepsilon \in W(0,T)$ where $\|y_\varepsilon\|$ is bounded by $2\|y\|$ according to (7.11) and the inverse triangle equation. Lemma 7.2.2 provides*

$$\left(\int_0^T |D_t y_\varepsilon|^2\right)^{1/2} \leq \tilde{c} + \left(\int\limits_0^T (f + u - \mathbf{b}\cdot\nabla y_\varepsilon(t))^2 dt\right)^{1/2} \leq c$$

where the constant c depends, besides the data f and initial condition u_0, on $|u|$ and $\|y\|$. Consequently, the generalized time derivatives $D_t y_\varepsilon$ are bounded in $\mathfrak{H}$ and have a weakly converging subsequence in this space. By Proposition 2.1.2 the limit element is the generalized time derivative of y thus being in $\mathfrak{H}$.
In addition, Theorem A.4.3 yields $|D_t y| \leq c$ with the same constant as above.

According to the regularity of solutions in the viscosity sense, established in Remark 7.2.4, one might claim, that the problem can be reduced to control functions u that with a representation of the form

$$u = D_t w + A^0 w + f \text{ for some } w \in W(0,T) \text{ satisfying } D_t w \in L^2(\mathcal{Q}). \tag{7.12}$$

We point out, that, similar to Example 6.2, one can construct feasible pairings (y,u) for the variational inequality problem (VI^T) where u can not be expressed by (7.12). As in Example 6.2, the differential operator A^0 has to be degenerated on a certain, open sub domain of Ω, i.e. the vector field $\mathbf{b}$ has to vanish there. Thus y solves the time dependent variational inequality problem for some u, admitting the claimed structure on the interior of $\mathcal{Q}$, excluded this sub domain times $[0,T]$.
By Remark 7.2.4 we can establish the following result similar to Theorem 4.1.3.

Proposition 7.2.2. *Consider a sequence of first order hyperbolic variational inequalities*

$$y_n \in \mathcal{K}, (D_t y_n + \mathbf{b}\cdot\nabla y_n + b^0 y_n - f - u_n, v - y_n) \geq 0 \textit{ for all } v \in \mathcal{K}, \;\; y_n(0) = u_0$$

such that $u_n \rightharpoonup u$ in $\mathfrak{H}$ and $y_n \in W(0,T)$.
If y_n is bounded in $\mathfrak{V}$ there exists a subsequence and a limit element $y \in W(0,T)$ with $y_n \rightharpoonup y$ such that y satisfies $D_t y \in \mathfrak{H}$ and

$$y \in \mathcal{K}, (D_t y + \mathbf{b}\cdot\nabla y + b^0 y - f - u, v - y) \geq 0 \textit{ for all } v \in \mathcal{K}, \;\; y(0) = u_0.$$

Proof. Let c denote the upper bound on the sequence y_n in $\mathfrak{V}$ and $\hat{c}$ the upper bound on $|u_n|$ which exist according to Theorem A.4.3.
Next we establish a bound on $\{D_t y_n\}$. By Theorem 7.2.3 for fixed n y_n can be approximated by a sequence y_{n_ε} of solutions to auxiliary parabolic variational inequalities with data u_n. All

those approximate solutions satisfy the following bound due to (7.11) and the inverse triangle inequality.

$$\|y_{n_\varepsilon}\| \leq 2\|y_n\| \leq 2c$$

Considering Remark 7.2.4 we in addition find the estimate

$$|D_t y_n| \leq c_n \leq \tilde{c}$$

where the constant c_n depends on $\|y_n\|, |u_n|, f$ and u_0. $\tilde{c}$ is obtained from $c, \hat{c}$ and independent of n. Thus the generalized time derivatives of y_n with respect to t are bounded in $\mathfrak{H}$ independent of n and we find

$$y_n \rightharpoonup y \text{ in } W(0,T)$$

for some limit element $y \in W(0,T)$ with $D_t y \in \mathfrak{H}$ (see Theorem A.4.3 and Proposition 2.1.2). The set $\mathcal{K}$ is weakly closed and thus $y \in \mathcal{K}$ has to hold by the weak convergence of y_n in $\mathfrak{V}$. Since we have shown that $D_t y_n$ converges weakly in $\mathfrak{H}$ and u_n and $\mathbf{b} \cdot \nabla y_n + b^0 y_n$ converge weakly in $\mathfrak{H}$ by the given regularity, we find by the compact embedding $W(0,T) \to \mathfrak{H}$, that

$$\lim_{n\to\infty} (D_t y_n + \mathbf{b} \cdot \nabla y_n + b^0 y_n - f - u_n, v - y_n) = (D_t y + \mathbf{b} \cdot \nabla y + b^0 y - f - u, v - y) \geq 0$$

holds along a further subsequence again denoted by n for all $v \in \mathcal{K}$. The prove, that the initial condition is met follows as in Lemma 7.2.5 and is omitted. □

Note the similarity of the preceding result to Theorem 4.1.3. A sequence of solutions in the viscosity sense to (VI^T) which can be bounded in a certain way provides weak convergence to a further solution in the viscosity sense of the variational inequality. This is a perfect transition of the theorem regarding classical viscosity solutions to our context. Besides the fact, that the solution we are considering are constructed utilizing a vanishing viscosity technique, this is a further property justifying the name of this class of solutions.
Lemma 7.2.5 provides the existence of solutions in the sense of Definition 7.2.2 for convex sets $\mathcal{K}$ constructed by $\mathbf{K}_2$ and $\mathbf{K}_3$ automatically since a bound on $\|y_\varepsilon\|$ independent of ε is introduced by the nature of the sets $\mathbf{K}_2, \mathbf{K}_3$. Unfortunately, this is not the case for sets of the form $\mathbf{K}_1$.

7.3 The Problems of Optimal Control

From this point on we will focus on the obstacle problem, i.e. sets $\mathcal{K}$ constructed by sets of the type $\mathbf{K} = \mathbf{K}_1$. In this section we establish the existence of solutions to the problem

$$\begin{array}{llll} \min & & \frac{1}{2}|y-y^d|^2 + \frac{\tilde{\beta}}{2}\|y\|^2 + \frac{\beta}{2}|u| & = \mathcal{J}(y,u) \\ \text{s.t.} & y \in \mathcal{K}, & \langle D_t y + A^0 y - f - u, v - y\rangle & \geq 0 \quad \forall v \in \mathcal{K} \\ & & y(0) & = u_0 \in H_0^1(\Omega) \cap H^2(\Omega) \cap \mathbf{K} \end{array} \qquad (\tilde{P}^T)$$

as well as for the approximating parabolic problems

$$\begin{array}{llll} \min & & \frac{1}{2}|y-y^d|^2 + \frac{\tilde{\beta}}{2}\|y\|^2 + \frac{\beta}{2}|u| & = \mathcal{J}(y,u) \\ \text{s.t.} & y \in \mathcal{K}, & \langle D_t y + A^\varepsilon y - f - u, v - y\rangle & \geq 0 \quad \forall v \in \mathcal{K} \\ & & y(0) & = u_0 \in H_0^1(\Omega) \cap H^2(\Omega) \cap \mathbf{K} \end{array} \qquad (\tilde{P}^T_\varepsilon)$$

7.3.1 The Problem subject to a First Order VI

Proposition 7.3.1. *Problem* $(\tilde{P}^T)$ *admits a solution.*

Proof. We consider the feasible set

$$\mathcal{D} = \{(y,u) \in W(0,T) \times \mathfrak{H} : y \in \mathcal{K}, \langle D_t y + \mathbf{b}\cdot\nabla y + b^0 y - f - u, v - y\rangle \geq 0, y(0) = u_0\}$$

of $(\tilde{P}^T)$. By the assumed regularity of u_0, the choice $\hat{y}(x,t) = u_0(x)$ and $\hat{u} = \mathbf{b}\cdot\nabla u_0 + b^0 u_0 - f$ is feasible and consequently $\mathcal{D} \neq \emptyset$. Next consider an infimizing sequence $\{(y_n, u_n)\} \in \mathcal{D}$ with

$$\lim_{n\to\infty} \mathcal{J}(y_n, u_n) = \inf_{(y,u)\in\mathcal{D}} \mathcal{J}(y,u) = d.$$

Due to the norm character of $\mathcal{J}$ we find $d \geq 0$. Since the objective is coercive, we obtain

$$\|y_n\| \leq ((2/\tilde{\beta})|\hat{y} - y^d|^2 + \tilde{\beta}/2\|\hat{y}\|^2 + \beta/2|\hat{u}|)^{1/2} = c$$

and $|u_n| \leq c$ for all n. Consequently, u_n contains a weakly convergent subsequence denoted the same with a corresponding limit element $\tilde{u} \in \mathfrak{H}$. Proposition 7.2.2 provides the existence of a further subsequence such that y_n converges weakly in $W(0,T)$ to some $\tilde{y}$ solving (VI^T) for $\tilde{u}$. $\mathcal{K}$ is closed and bounded implying, that limits of weakly converging sequences of elements are an element of $\mathcal{K}$ as well. Thus, $(\tilde{y}, \tilde{u})$ is feasible for $(\tilde{P}^T)$. The weak lower semicontinuity of $\mathcal{J}$ provides

$$d = \liminf_{n\to\infty} \mathcal{J}(y_n, u_n) \geq \mathcal{J}(\tilde{y}, \tilde{u}) \geq d$$

ensuring, that the infimum is attained. □

For the reformulation of $(\tilde{P}^T)$ we prove the following result similar to [7]. We repeat the proof to show, that the nature of the differential operator does not effect the result.

Lemma 7.3.1. *Consider*

$$y \in \mathcal{K}, \langle D_t y + A^0 y - f - u, v - y\rangle \geq 0 \textit{ for all } v \in \mathcal{K}, y(0) = u_0$$

for $f \in \mathfrak{H}$ *and* $u \in \mathfrak{H}$*. If* y *is a solution in the viscosity sense of this problem, the variational inequality is equivalent to the complementarity system*

$$D_t y + A^0 y - f - u = \xi, \;\; \xi \geq 0, \;\; y \geq \psi, \;\; (\xi, y - \psi) = 0, \;\; y(0) = u_0$$

with $\xi \in \mathfrak{H}$*.*

Proof. According to Remark 7.2.4, $D_t y \in \mathfrak{H}$, and $f, u, A^0 y \in \mathfrak{H}$ by regularity. Thus $\xi \in \mathfrak{H}$ has to hold allowing for a pointwise interpretation. From the variational inequality we obtain

$$(\xi, v - y) \geq 0 \text{ for all } v \in \mathcal{K} \Leftrightarrow (\xi(t), v - y(t))_{L^2(\Omega)} \geq 0 \text{ for all } v \in \mathbf{K} \text{ for a. e. } t \in (0,T)$$

by Lemma 7.3.1. For every $\eta \in H_0^1(\Omega)$ satisfying $\eta \geq 0$ almost everywhere we find for $v = y(t) + \eta$ the identity

$$(\xi(t), v - y(t)) = (\xi(t), \eta) \geq 0.$$

By density, $(\xi(t), \eta) \geq 0$ can be extended to all $\eta \in L^2(\Omega)$ with $\eta \geq 0$ implying, that $\xi(t)$ is nonegative almost everywhere in $\mathcal{Q}$. Using $v = \psi$ and $v = 2y(t) - \psi$ we get

$$(\xi(t), y(t) - \psi)_{L^2(\Omega)} = 0 \text{ for a.e. } t \in (0,T) \quad \Rightarrow \quad (\xi, y - \psi) = 0.$$

The initial condition is not effected by the reformulation. Moreover, $y \geq \psi$ holds a.e. in $\mathcal{Q}$ since $y \in \mathcal{K}$. Consequently the solution satisfies the complementarity system.
The converse direction follows from $(\xi, y) = 0$ and $(\xi, v) \geq 0$ for all $v \in \mathcal{K}$. □

Consequently we can reformulate $(\tilde{P}^T)$ and obtain the equivalent problem.

$$\begin{array}{lrl} \min & \frac{1}{2}|y - y^d|^2 + \frac{\tilde{\beta}}{2}\|y\|^2 + \frac{\beta}{2}|u| & = \mathcal{J}(y,u) \\ \text{s.t.} & D_t y + \mathbf{b} \cdot \nabla y + b^0 y - f - u & = \xi \\ & (\xi, y - \psi) & = 0 \\ & y - \psi \geq 0, \qquad \xi & \geq 0 \text{ a.e. in } \mathcal{Q} \\ & y(0) & = u_0 \in H_0^1(\Omega) \cap H^2(\Omega) \cap \mathbf{K} \end{array} \tag{P^T}$$

7.3.2 The Family of Approximating Problems

We start with establishing a result concerning bounded sequences of solutions for (VI_ε^T) for fixed $0 < \varepsilon \leq \bar{\varepsilon}$.

Lemma 7.3.2. *Consider a sequence $\{y_n, u_n\}$ bounded in $W(0,T) \times L^2(\mathcal{Q})$ and satisfying (VI_ε^T) for a fixed initial condition $u_0 \in V \cap \mathbf{K}$.*
Then there exist weak accumulation points $(\tilde{y}, \tilde{u}) \in W(0,T) \times \mathfrak{H}$ with

$$y_n \rightharpoonup \tilde{y} \text{ in } W(0,T) \text{ and } u_n \rightharpoonup \tilde{u} \text{ in } \mathfrak{H}.$$

In addition the limit points solve (VI_ε^T) for $u_0 \in V \cap \mathbf{K}$.

Proof. The assumed boundedness yields the existence of the weak accumulation points

$$(\tilde{y}, \tilde{u}) \in W(0,T) \times \mathfrak{H}.$$

The weak closeness of $\mathcal{K}$ ensures $\tilde{y} \in \mathcal{K}$. Since every pairing solves (VI_ε^T) it also satisfies

$$y_n \in \mathcal{K} \quad \langle D_t v + A^\varepsilon y_n - f - u_n, v - y_n \rangle_{\mathfrak{V}^*,\mathfrak{V}} \geq 0 \text{ for all } v \in \mathcal{K} \cap W(0,T) \text{ with } v(0) = u_0$$

according to Lemma 7.1.1. Adding a zero and rearranging the terms yields

$$\langle D_t v + A^\varepsilon y_n - f - u_n, v - \tilde{y} \rangle_{\mathfrak{V}^*,\mathfrak{V}} \geq \langle D_t v + A^\varepsilon y_n - f - u_n, y_n - \tilde{y} \rangle_{\mathfrak{V}^*,\mathfrak{V}} \tag{7.13}$$

By $y_n \rightharpoonup \tilde{y}$ in $\mathfrak{V}$ we obtain $\langle D_t v, y_n - \tilde{y} \rangle_{\mathfrak{V}^*,\mathfrak{V}} \to 0$. Since we find a subsequence strongly converging in $L^2(\mathcal{Q})$ we further get $(f - u_n, y_n - \tilde{y}) \to 0$. By Remark 7.2.3, Lemma 7.2.2, (7.8) and the inverse triangle inequality we obtain boundedness of $A^\varepsilon y_n$ in $\mathfrak{H}$ and the existence of a weak accumulation point $\Theta \in \mathfrak{H}$. Consequently $\langle A^\varepsilon y_n - f - u_n, y_n - \tilde{y} \rangle_{\mathfrak{V}^*,\mathfrak{V}} \to (\Theta, 0) = 0$ holds. Thus the limit for $n \to \infty$ in(7.13) yields

$$\langle D_t v + A^\varepsilon \tilde{y} - \tilde{f} - \tilde{u}, v - \tilde{y} \rangle_{\mathfrak{V}^*,\mathfrak{V}} \geq 0 \text{ for all } v \in W(0,T) \cap \mathcal{K} \text{ with } v(0) = u_0.$$

Consequently, $\tilde{y}$ is a weak solution of (VI_ε^T) for $\tilde{u}$ which is unique according to Theorem 7.1.1. By Theorem 7.2.1, there also exist a unique solution $\hat{y} \in W(0,T)$ to (VI_ε^T) as well and thus

$\hat{y} = \tilde{y}$ has to hold according to Lemma 7.1.1. □

Next we consider $(\tilde{P}_\varepsilon^T)$. The solvability of this problem is shown by standard arguments as for example presented in [95].

Proposition 7.3.2. *$(\tilde{P}_\varepsilon^T)$ attains a solution.*

Proof. Let $u_0 \in \mathbf{K} \cap H_0^1(\Omega) \cap H^2(\Omega)$ be given and let

$$\mathcal{D} = \{(y,u)|(y,u) \text{ satisfies } (VI_\varepsilon^T)\}$$

denote the feasible set of $(\tilde{P}_\varepsilon^T)$. By the regularity of $u_0 \in \mathbf{K}$, the pairing $(\hat{y},\hat{u})$ with $\hat{y}(x,t) = u_0(x)$ and $\hat{u}(x,t) = -f(x,t) + A^\varepsilon u_0(x,t)$ is feasible. Thus $\mathcal{D}$ is not empty. Due to the norm character, $\mathcal{J}(y,u)$ is bounded from below. Next, consider an infimizing sequence $\{(y_n,u_n)\}$ with $y_n = y(u_n)$ denoting the solution of (VI_ε^T) for u_n and

$$\lim_{n\to\infty} \mathcal{J}(y_n,u_n) = \inf_{(y,u)\in\mathcal{D}} \mathcal{J}(y,u) = d \geq 0.$$

$\mathcal{J}$ is coercive implying the boundedness of $\|y_n\|$, and $|u_n|$. By Lemma 7.2.2, $\{D_t y_n\}$ is bounded in $\mathfrak{H}$ providing the boundedness of $\{y_n\}$ in $W(0,T)$. Hence, there exist weak accumulation points $(\tilde{y},\tilde{u}) \in W(0,T) \times \mathfrak{H}$ such that

$$y_n \rightharpoonup \tilde{y} \text{ in } W(0,T) \quad \text{and } u_m \rightharpoonup \tilde{u} \text{ in } \mathfrak{H}$$

along a subsequence denoted by the same indexes. By Lemma 7.3.2, those points are a solution to (VI_ε^T). Since $\mathcal{J}$ is weakly lower semi continuous, we find

$$d = \liminf_{n\to\infty} \mathcal{J}(y_n,u_n) \geq \mathcal{J}(\tilde{y},\tilde{u}) \geq d.$$

Consequently the infimum of $\mathcal{J}$ is attained and $(\tilde{y},\tilde{u})$ is optimal for $(\tilde{P}_\varepsilon^T)$. □

At this point the regularity requirement on u_0 becomes obvious since it is needed for the construction of feasible points of $(\tilde{P}_\varepsilon^T)$. It is well known, that the variational inequality (VI_ε^T) is equivalent to a certain complementarity system (cf [7]). Thus $(\tilde{P}_\varepsilon^T)$ is equivalent to

$$\begin{array}{lrl} \min & \frac{1}{2}|y-y^d|^2 + \frac{\tilde{\beta}}{2}\|y\|^2 + \frac{\beta}{2}|u|^2 & \\ \text{s.t.} & D_t y - \varepsilon\Delta y + \mathbf{b}\cdot\nabla y + b^0 y - f - u + \xi & = 0 \\ & (\xi, y-\psi)_{L^2(\mathcal{Q})} & = 0 \\ & y - \psi \geq 0 \qquad \xi & \geq 0 \text{ a.e. in } \mathcal{Q} \\ & y(0) & = u_0 \text{ a.e. in } \Omega \end{array} \tag{P_ε^T}$$

Note, that by Theorem 7.2.1 the introduced variable ξ is an element of $\mathfrak{H}$ and hence admits a pointwise interpretation a.e.. We finally present a consistency result for the approximating problems justifying their introduction.

Proposition 7.3.3. *Let $\varepsilon \to 0$ be a sequence of viscosity parameters with $0 < \varepsilon \leq \bar{\varepsilon}$. Consider a sequence $(y_\varepsilon^*, u_\varepsilon^*)$ of solutions to the corresponding problems $(\tilde{P}_\varepsilon^T)$. Then there exist a subsequence of the parameters such that*

$$y_\varepsilon^* \rightharpoonup \tilde{y} \text{ in } W(0,T) \quad \text{and} \quad u_\varepsilon^* \rightharpoonup \tilde{u} \text{ in } L^2(\mathcal{Q})$$

along this subsequence. Moreover, the weak accumulation points $(\tilde{y}, \tilde{u})$ are optimal for $(\tilde{P}^T)$.

Proof. For any ε we find the feasible point

$$\hat{y}(x,t) = u_0(x) \text{ and } \hat{u}_\varepsilon(x,t) = -\varepsilon\Delta u_0(x) + \mathbf{b}\cdot\nabla u_0(x) + b^0 u_0(x) - f(x,t)$$

where $\hat{u}_\varepsilon \in \mathfrak{H}$ be the regularity assumptions on the initial condition. Now we estimate

$$|\hat{u}_\varepsilon| \leq \varepsilon|\Delta u_0| + |\mathbf{b}\cdot\nabla u_0 + b^0 u_0 - f| \leq \bar{\varepsilon}|\Delta u_0| + |\mathbf{b}\cdot\nabla u_0 + b^0 u_0 - f|$$

and find

$$\frac{\tilde{\beta}}{2}\|y_\varepsilon^*\|^2 + \frac{\beta}{2}|u_\varepsilon^*|^2 \leq \frac{1}{2}|\hat{y} - y^d|^2 + \frac{\tilde{\beta}}{2}\|\hat{y}\|^2 + \frac{\beta}{2}c^2$$

providing an upper bound $\|y_\varepsilon^*\|$ and $|u_\varepsilon^*|$ which is independent of ε. According to Lemma 7.2.5 there exist weakly converging subsequences and corresponding limit elements $\tilde{y}$ and $\tilde{u}$ with

$$y_\varepsilon^* \rightharpoonup \tilde{y} \text{ in } W(0,T) \quad \text{and} \quad u_\varepsilon^* \rightharpoonup \tilde{u} \text{ in } L^2(\mathcal{Q})$$

satisfying

$$\tilde{y} \in \mathcal{K},\ (D_t\tilde{y} + \mathbf{b}\cdot\nabla\tilde{y} + b^0\tilde{y} - f - \tilde{u}, v - \tilde{y}) \geq 0 \text{ for all } v \in \mathcal{K}, \tilde{y}(0) = u_0.$$

Consequently, the weak limit point is feasible for $(\tilde{P}^T)$. In order to proof the optimality we consider an arbitrary feasible pairing $(y,u) \in W(0,T)\times L^2(\mathcal{Q})$ for problem $(\tilde{P}^T)$. By Theorem 7.2.3 we find approximating solutions $y_\varepsilon \in W(0,T)$ of the variational inequalities (VI_ε^T) for the given control u with

$$y_\varepsilon \rightharpoonup y \text{ in } W(0,T) \quad \text{and} \quad y_\varepsilon \to y \text{ in } \mathfrak{V}.$$

We highlight the strong convergence in the latter space. Thus we obtain

$$\mathcal{J}(y,u) = \lim_{\varepsilon\to 0}\mathcal{J}(y_\varepsilon, u_\varepsilon) \geq \liminf_{\varepsilon\to 0}\mathcal{J}(y_\varepsilon^*, u_\varepsilon^*) \geq \mathcal{J}(\tilde{y},\tilde{u})$$

where the first estimate stems from the optimality of $(y_\varepsilon^*, u_\varepsilon^*)$ for problem $(\tilde{P}_\varepsilon^T)$ and the second from the weak lower semicontinuity of the objective $\mathcal{J}$. Consequently, $(\tilde{y},\tilde{u})$ is optimal for the original problem since no other feasible point performs better in the objective. □

Remark 7.3.1. *By the already proven equivalence in Lemma 7.3.1, this consistency property carries over to (P_ε^T) and (P^T). Note that the slack variables weakly converge in $\mathfrak{H}$ instead of $\mathfrak{V}^*$ for the following reason. The state equation for the MPCC yields $\xi_\varepsilon = D_t y_\varepsilon + A^\varepsilon y_\varepsilon - u_\varepsilon - f$. Now* (7.8) *provides*

$$|\xi_\varepsilon| = |D_t y_\varepsilon + A^\varepsilon y_\varepsilon - u_\varepsilon - f| \leq |D_t y_\varepsilon + A^\varepsilon y_\varepsilon| + |u_\varepsilon + f| \leq c$$

by the boundedness of u_ε in $L^2(Q)$.

7.4 Stationarity

This section deals with the procedure for obtaining necessary optimality conditions for problem $(\tilde{P}^T)$ or equivalently (P^T). It consist of three basic steps. First we use the approximating problems (P_ε^T) and introduce a parametrized sequence of approximating problems by penalizing the violation of $y \geq \psi$ and regularizing the condition for the product. Then we derive necessary first order optimality conditions for the nonlinear problems using classical theory. In the second step we discuss the limit of the corresponding system when the parameters are driven their limits. At this stage, the viscosity parameter ε is still fixed. Finally we analyze the behavior of the resulting system if the viscosity parameter vanishes providing a necessary optimality condition for the original problem.

The results of Section 7.4.1 and 7.4.2 are slight modifications of the theory developed in [95]. Since the reference is hard to find we will present parts of some proofs and omit the ones where mostly standard arguments are used only outlining the main idea.

7.4.1 Penalization and Regularization of the Approximating Problems

In order to derive a first order stationary system for $(\tilde{P}_\varepsilon^T)$ we again utilize the penalization-regularization technique from Chapter 6 and obtain the following optimization problem subject to a partial differential equation with additional constraints.

$$\begin{array}{lrl} \min & \frac{1}{2}|y-y^d|^2 + \frac{\tilde{\beta}}{2}\|y\|^2 + \frac{\beta}{2}|u|^2 + \frac{1}{2\gamma}|(\bar{\lambda}-\gamma(y-\psi))^+|^2 + \frac{\kappa}{2}|\xi|^2 & = \tilde{\mathcal{J}}_\gamma \\ \text{s.t.} & D_t y - \varepsilon\Delta y + \mathbf{b}\cdot\nabla y + b^0 y - f - u + \xi = 0 & \text{in } \mathcal{Q} \\ & y(0) = u_0 & \text{in } \Omega \\ & \xi \geq 0 & \text{in } \mathcal{Q} \\ & (\xi, y-\psi)_{L^2(\mathcal{Q})} \leq \alpha & \end{array} \qquad (P^T_{\varepsilon_{\gamma,\alpha,\kappa}})$$

The following result can be found in [95].

Proposition 7.4.1. *Let $(\alpha,\kappa,\gamma) > 0$ be fixed. Then problem $(P^T_{\varepsilon_{\gamma,\alpha,\kappa}})$ has at least one globally optimal solution.*

The proof establishes the boundedness of an infimizing sequence of elements in the feasible set of $(P^T_{\varepsilon_{\gamma,\alpha,\kappa}})$ in $W(0,T)\times\mathfrak{H}\times\mathfrak{H}$ utilizing the coercivity of the objective functional and regularity theory for parabolic partial differential equations. Due to Theorem A.4.2 and certain imbedding properties of the involved spaces, a weakly converging subsequence can be found such that the weak limit points are feasible for $(P^T_{\varepsilon_{\gamma,\alpha,\kappa}})$ as well. The optimality of this point is shown by the weak lower continuity of the objective.

After establishing the existence of solutions for problem $(P^T_{\varepsilon_{\gamma,\alpha,\kappa}})$ we focus on the characterization of such points. The natural approach is to formulate a KKT system in the sense of Theorem 3.1.2.

Proposition 7.4.2. *Let $(\bar{y},\bar{u},\bar{\xi}) \in W(0,T)\times\mathfrak{H}\times\mathfrak{H}$ be an optimal solution of $(P^T_{\varepsilon_{\gamma,\alpha,\kappa}})$ for $(\alpha,\kappa,\gamma) > 0$. Then there exist Lagrange multipliers $(p,\mu,\nu) \in W(0,T)\times L^2(\mathcal{Q})\times\mathbb{R}^+$ such* *that the following system is satisfied.*

$$-D_t p + (A^\varepsilon)^* p + (\bar y - y^d) - \tilde\beta \Delta \bar y - (\bar\lambda - \gamma(\bar y - \psi))^+ + \nu\bar\xi = 0 \tag{7.14a}$$
$$p(x,t) = 0 \text{ for } x \in \partial\Omega, \quad p(T) = 0 \tag{7.14b}$$
$$\beta\bar u - p = 0 \tag{7.14c}$$
$$\kappa\bar\xi - p - \mu + \nu(\bar y - \psi) = 0 \tag{7.14d}$$
$$\bar\xi \geq 0 \text{ a.e.} \quad \mu \geq 0 \text{ a.e.} \quad (\bar\xi, \mu)_{L^2(\mathcal{Q})} = 0 \tag{7.14e}$$
$$\alpha \geq (\bar\xi, \bar y - \psi)_{L^2(\mathcal{Q})} \text{ a.e.} \quad \nu \geq 0 \qquad \nu(\alpha - (\bar\xi, \bar y - \psi)_{L^2(\mathcal{Q})}) = 0 \tag{7.14f}$$
$$D_t \bar y + A^\varepsilon \bar y - f - \bar u - \bar\xi = 0 \tag{7.14g}$$
$$\bar y(x,t) = 0 \text{ for } x \in \partial\Omega \quad \bar y(0) = u_0 \tag{7.14h}$$

Proof. The mapping $g : W(0,T) \times \mathfrak{H} \times \mathfrak{H} \to \mathfrak{V}^* \times L^2(\Omega) \times \mathfrak{H} \times \mathbb{R}$ defines the constraints of $(P^T_{\varepsilon_\gamma,\alpha,\kappa})$ by

$$g(y,u,\xi,\alpha) = (D_t y + A^\varepsilon y - f - u + \xi, \mathcal{R}_0 y - u_0, \xi, \alpha - (\xi, y - \psi))$$

Here $\mathcal{R}_0 : W(0,T) \to L^2(\Omega)$ is the linear restriction operator $\mathcal{R}_0 v = v(0,\cdot)$ well defined by the continuous embedding $W(0,T) \to C([0,T]; L^2(\Omega))$. Moreover we define the cone $\hat K = \{0\} \times \{0\} \times L^2(\mathcal{Q})_+ \times \mathbb{R}_+$. One can prove, that the constraint qualifications of Theorem 3.1.2 are met and thus the set of Lagrange multipliers is nonempty and bounded. So there exist $(p, \rho, \mu, \nu) \in Y^* = \mathfrak{V} \times L^2(\Omega) \times L^2(\mathcal{Q}) \times \mathbb{R}$ with $\mu \geq 0$ a.e. and $\nu \geq 0$. Moreover we have

$$(\mu, \xi) = 0, \quad \nu(\alpha - (\xi, \psi - y)) = 0 \quad \text{and} \quad J'_\gamma(\bar x) - \tilde y g'(\bar x) = 0$$

Due to the third point we obtain

$$\bar y - y^d - \tilde\beta\Delta\bar y - (\bar\lambda - \gamma(\bar y - \psi))^+ + (D_t + A^\varepsilon)^* p - \mathcal{R}_0^*\rho + \nu\bar\xi = 0 \tag{7.15}$$
$$\beta\bar u - p = 0 \tag{7.16}$$
$$\kappa\xi - p - \mu + \nu(\bar y - \psi) = 0 \tag{7.17}$$

where (7.16) and (7.17) provide (7.14c) and (7.14d). Since in the following part the different nature of the adjoint mapping has to be considered, we repeat the part of the proof from [95] for our setting. Introducing the abbreviation

$$\Theta = \bar y - y^d - \tilde\beta\Delta\bar y - (\bar\lambda - \gamma(\bar y - \psi))^+ + \nu\bar\xi \in \mathfrak{V}^*$$

we get from (7.15) by testing with an arbitrary function $w \in W(0,T)$

$$\langle \Theta, w\rangle_{\mathfrak{V}^*,\mathfrak{V}} + \langle D_t w, p\rangle_{\mathfrak{V}^*,\mathfrak{V}} + \langle A^\varepsilon w, p\rangle_{\mathfrak{V}^*,\mathfrak{V}} - (\rho, \mathcal{R}_0 w)_H = 0 \tag{7.18}$$

For the special choice of $w(t,x) = \eta(t)v(x)$ with $\eta \in C_c^\infty(0,T)$ and $v \in H_0^1(\Omega)$ this yields

$$\int_0^T \eta(t)\langle\Theta(t), v\rangle dt + \int_0^T \eta'(t)\langle p(t), v\rangle dt + \int_0^T \eta(t)\langle (A^\varepsilon)^* p(t), v\rangle dt = 0$$

which is equivalent to

$$\int_0^T \eta'(t)(p(t), v)dt = -\int_0^T \eta(t)\langle\Theta(t) + (A^\varepsilon)^*p(t), v\rangle dt.$$

As v and η were chosen arbitrary we obtain obtain from Definition 2.1.1, that the distributional derivative of p is given via $\Theta + (A^\varepsilon)^*p \in \mathfrak{V}^*$. Consequently, p is an element of $W(0,T)$ and we obtain $-D_tp + \Theta + (A^\varepsilon)^*p = 0$ Testing this with some $w \in W(0,T)$ and integrating by parts yields

$$\langle\Theta, w\rangle + \langle A^\varepsilon w, p\rangle - (p(T), w(T))_{L^2(\Omega)} + (p(0), w(0))_{L^2(\Omega)} + \langle D_t w, p\rangle = 0. \tag{7.19}$$

Subtracting (7.18) from (7.19) and testing with $w \in W(0,T)$ satisfying either $w(0) = 0$ or $w(T) = 0$ yield

$$p(0) = -\rho \quad \text{ and } p(T) = 0.$$

This provides (7.14b). □

7.4.2 Limit of the Relaxed-Regularized Auxiliary Problems

In this part we will establish a consistency result for the problems $(P^T_{\varepsilon_{\gamma,\alpha,\kappa}})$ and prove the existence of a limiting stationarity system. Moreover, we will present an update rule for the involved parameters in the following results. Assume $\gamma \geq \hat{\gamma}$.

Theorem 7.4.1. *For every $\gamma > 0$ let $(\alpha_\gamma, \kappa_\gamma) > 0$ be given, such that $(\alpha_\gamma, \kappa_\gamma) \to 0$ for $\gamma \to \infty$. Further, let $(y_\gamma, u_\gamma, \xi_\gamma)$ be a solution of $(P^T_{\varepsilon_{\gamma,\alpha,\kappa}})$. Then $\{y_\gamma\}$ is bounded in $C([0,T]; L^2(\Omega))$. Moreover, there exist $(\tilde{y}, \tilde{u}) \in \mathfrak{V} \times \mathfrak{H}$ such that*

$$y_\gamma \rightharpoonup \tilde{y} \text{ in } \mathfrak{V}, u_\gamma \rightharpoonup \tilde{u} \text{ in } \mathfrak{H}$$

If in addition the assumption

$$\lim_{\gamma\to\infty}(y_\gamma, u_\gamma) = (\tilde{y}, \tilde{u})$$

is satisfied, then $\tilde{y} \in W(0,T)$ and there exist $\tilde{\xi} \in L^2(\mathcal{Q})$ such that $(\tilde{y}, \tilde{u}, \tilde{\xi})$ is a solution of (P^T_ε) and we have

$$y_\gamma \to \tilde{y} \text{ in } \mathfrak{V}$$

Again the proof is largely similar to the result presented in [95] up to some slight modifications except one point. Since the named reference only considers symmetric coercive differential operators,

$$\langle\mathcal{A}\cdot, \cdot\rangle_{\mathfrak{V}^*,\mathfrak{V}}$$

is a positive definite and symmetric bilinear form on $\mathfrak{V}$ and thus automatically weakly lower semicontinuous. For general nonsymmetric operators this does not hold. However, $\langle A^\varepsilon y_\gamma, y_\gamma\rangle$ is weakly lower semicontinuous from $\mathfrak{V}$ to $\mathbb{R}$ by Remark 7.2.1 and the proof still works.
By the theorem we have shown, that the relaxed regularized problems are consistent with the parabolic auxiliary problems. This consistency result only holds true if the parameters are updated in a suitable coupled way. Thus, we require the following.

Assumption 7.1. *Let the update strategy for the parameters* $\gamma, \alpha_\gamma, \kappa_\gamma$ *satisfy the following conditions.*

$$(\alpha_\gamma, \kappa_\gamma) \to (0,0) \text{ for } \gamma \to \infty \text{ and } \max\{(\alpha_\gamma\sqrt{\gamma})^{-1}, \kappa_\gamma\sqrt{\gamma}\} \leq c$$

Next we want to derive a stationarity system for the limiting problem. Therefore we need the following additional assumption.

Assumption 7.2. *Let the sequence* $\{\xi_\gamma\}$ *be bounded in* $L^2(\mathcal{Q})$*. Moreover we assume that* $\{u_\gamma\}$ *stays bounded in* $L^2(\mathcal{Q})$*.*

Note that any sequence of minimizers for the parabolic problems satisfies Assumption 7.2. Thus it can be satisfied. We introduce the abbreviation $\lambda_\gamma = (\bar{\lambda} - \gamma(y_\gamma - \psi))$ and define

$$\begin{aligned} \Lambda_\gamma &= \{(x,t) \in \mathcal{Q} | \bar{\lambda}(x,t) - \gamma(y_\gamma(x,t) - \psi(x,t)) > 0\} \\ \mathcal{I}_\gamma &= \{(x,t) \in \mathcal{Q} | y_\gamma(x,t) - \psi(x,t) < 0\} \end{aligned}$$

Finally we present the stationarity system for the auxiliary problems in the following result.

Theorem 7.4.2. *For any* $\gamma > 0$ *let* $(\alpha_\gamma, \kappa_\gamma) > 0$ *be given according to Assumption 7.1. Further, let for any* $\gamma > 0$

$$(y_\gamma, u_\gamma, \xi_\gamma, p_\gamma, \nu_\gamma, \mu_\gamma) \in W(0,T) \times \mathfrak{H} \times \mathfrak{H} \times W(0,T) \times \mathbb{R}^+ \times \mathfrak{H}$$

satisfy the optimality system (7.14)*. Under Assumption 7.2 there exist*

$$(\tilde{y}, \tilde{u}, \tilde{\xi}, \tilde{p}, \tilde{\lambda}) \in W(0,T) \times \mathfrak{V} \times \mathfrak{H} \times \mathfrak{V} \times W(0,T)^*$$

and a subsequence, again denoted by $\{(y_\gamma, , u_\gamma, \xi_\gamma, p_\gamma, \nu_\gamma, \mu_\gamma)\}$ *satisfying*

$$\begin{gathered} y_\gamma \rightharpoonup \tilde{y} \text{ in } W(0,T) \text{ and strongly in } \mathfrak{V},\ u_\gamma \rightharpoonup \tilde{u} \text{ in } \mathfrak{V}, \xi_\gamma \rightharpoonup \tilde{\xi} \text{ in } \mathfrak{H} \\ p_\gamma \rightharpoonup \tilde{p} \text{ in } \mathfrak{V} \text{ and } (\lambda_\gamma - \nu_\gamma \xi_\gamma) \rightharpoonup \tilde{\lambda} \text{ in } W(0,T)^*. \end{gathered}$$

The limit elements satisfy the following system.

$$(\tilde{y} - y^d, v) - \tilde{\beta}\langle \Delta\tilde{y}, v\rangle - \langle\tilde{\lambda}, v\rangle_{W^*,W} + \langle\tilde{p}, (D_t + A^\varepsilon)v\rangle = 0 \ \ \forall v \in W_0(0,T) \tag{7.20a}$$

$$\beta\tilde{u} - \tilde{p} = 0 \tag{7.20b}$$

$$\tilde{\xi} \geq 0 \text{ a.e. } , \tilde{y} - \psi \geq 0 \text{ a.e. } , (\tilde{\xi}, \tilde{y} - \psi) = 0 \tag{7.20c}$$

$$D_t\tilde{y} + A^\varepsilon\tilde{y} - \tilde{u} - \tilde{\xi} = f \text{ in } \mathfrak{V}^* \tag{7.20d}$$

$$\tilde{y}(0) = u_0 \tag{7.20e}$$

$$\lim_{\gamma\to\infty} \langle\lambda_\gamma - \nu_\gamma\xi_\gamma, (y_\gamma - \psi)^+\rangle_{\mathfrak{V}^*,\mathfrak{V}} = 0 \tag{7.20f}$$

$$\lim_{\gamma\to\infty} (p_\gamma, \xi_\gamma)_{L^2(\omega)} = 0 \forall\omega \subset \mathcal{Q} \tag{7.20g}$$

Here $W_0(0,T) = \{\varphi \in W(0,T) : \varphi(0) = 0\}$ *is a closed linear subspace of* $W(0,T)$*.*
Moreover, for every $\tau > 0$*, there exist a subset* $E_+ \subset \mathcal{Q}_+ = \{(x,t) \in \mathcal{Q} | \tilde{y}(x,t) > \psi(x,t)\}$ *with* $|\mathcal{Q}_+ \backslash E_+| \leq \tau$ *and*

$$(\lambda_\gamma - \nu_\gamma\xi_\gamma) \to 0 \text{ uniformly on } E_+. \tag{7.21}$$

If the updating of κ_γ even satisfies $\kappa_\gamma\sqrt{\gamma} \to 0$, then

$$\limsup_{\gamma\to\infty} \langle \lambda_\gamma - \nu_\gamma \xi_\gamma, -p_\gamma \rangle_{\mathfrak{Y}^*,\mathfrak{Y}} \le 0. \tag{7.22}$$

The proof is again almost as in [95]. In general, if the adjoint equation is considered in the original proof, an additional usage of Theorem A.4.8 is sufficient to proceed in the present case.

Remark 7.4.1. *The assumed regularity $u_0 \in H^2(\Omega)$ plays an important role for the existence of solutions of the involved problems. Besides assuming $u_0 \in H^2(\Omega)$, a further possibility to formulate the problem could be to consider u_0 as a control variable as well. Then $u_0 \in H_0^1(\Omega)$ would be sufficient since this is the minimal regularity required for an application of* (7.9)*. This can be ensured by an additional regularization term in the objective since the initial condition is naturally in $L^2(\Omega)$. The resulting problem is given by*

$$\begin{aligned} \textit{min} \quad & \tfrac{1}{2}|y-y^d|^2 + \tfrac{\tilde\beta}{2}\|y\|^2 + \tfrac{\hat\beta}{2}|u_0|^2_{H_0^1(\Omega)} + \tfrac{\beta}{2}|u|^2 \\ \textit{s.t.} \quad & D_t y + \mathbf{b}\cdot\nabla y + b^0 y - f - u + \xi = 0 \\ & (\xi, y-\psi)_{L^2(\mathcal{Q})} = 0 \\ & y-\psi \ge 0 \qquad \xi \ge 0 \ \textit{in } \mathcal{Q} \\ & y(0) = u_0 \ \textit{in } \Omega \end{aligned}$$

The existence of solutions can be shown similar to Proposition 7.3.2. Proving the existence of solutions to the approximating problems is possible as well but the first order necessary optimality condition would involve, besides a different adjoint, the additional equation

$$-\Delta u_0 = p_0$$

Here p_0 is the Lagrange multiplier for the constraint $\mathcal{R}_0(y) = u_0$ considered in $L^2(\Omega)$. By our assumptions on the domain Ω, u_0 would be an element of $H_0^1(\Omega)\cap H^2(\Omega)$ (see Theorem 2.2.4). Consequently, the regularity requirement we introduced is not a major restriction for the problem setting.

7.4.3 Limiting Stationarity Systems

In this section we establish the limiting system and hence stationarity system for (P^T).

Theorem 7.4.3. *Consider a sequence of viscosity parameters $\varepsilon \to 0$ converging to zero and corresponding elements $(y_\varepsilon, u_\varepsilon, \xi_\varepsilon, p_\varepsilon, \lambda_\varepsilon)$ satisfying* (7.20), (7.21) *and* (7.22)*.*
If the sequence $(y_\varepsilon, u_\varepsilon)$ is bounded in $\mathfrak{Y}\times L^2(\mathcal{Q})$ there exist weak limit points

$$(\tilde y, \tilde u, \tilde\xi, \tilde p, \tilde\lambda) \in \hat W(0,T)\times\mathfrak{H}\times\mathfrak{H}\times\mathfrak{H}\times(\hat W_0(0,T))^*$$

with

$$y_\varepsilon \rightharpoonup \tilde y \textit{ in } \hat W(0,T),\ \ u_\varepsilon \rightharpoonup \tilde u \textit{ in } \mathfrak{H},\ \ \xi_\varepsilon \rightharpoonup \tilde\xi \textit{ in } \mathfrak{H},\ \ p_\varepsilon \rightharpoonup \tilde p \textit{ in } \mathfrak{H},\ \ \lambda_\varepsilon + \varepsilon\Delta p_\varepsilon \rightharpoonup \tilde\lambda \textit{ in } \hat W_0(0,T)^*$$

satisfying the system

$$(\tilde{y} - y^d, \varphi) - \tilde{\beta}\langle \Delta \tilde{y}, \varphi\rangle + (\tilde{p}, (D_t + A^0)\varphi) - \langle \tilde{\lambda}, \varphi\rangle_{\hat{W}_0(0,T)^*, \hat{W}_0(0,T)} = 0 \tag{7.23a}$$

$$\beta \tilde{u} - \tilde{p} = 0 \tag{7.23b}$$

$$D_t \tilde{y} + A^0 \tilde{y} - f - \tilde{u} = \tilde{\xi} \tag{7.23c}$$

$$\tilde{y}(0) = u_0 \tag{7.23d}$$

$$\tilde{y} - \psi \geq 0,\ \tilde{\xi} \geq 0, (\tilde{y} - \psi, \tilde{\xi}) = 0 \tag{7.23e}$$

$$\textit{on } S = \{x \in \mathcal{Q} : \tilde{\xi}(x) > 0\} \textit{ we have the } \lim_{\varepsilon \to 0} [p_\varepsilon, \xi_\varepsilon]_{L^2(S)} = 0 \tag{7.23f}$$

$$\forall \tau > 0\ \exists \mathcal{Q}^\tau : |\mathcal{Q}_+ \backslash \mathcal{Q}^\tau| \leq \tau$$

$$\textit{and } \forall \varphi \in C_c^\infty(\mathcal{Q}) \textit{ with } \varphi(x) = 0 \textit{ for all } x \in \mathcal{Q}\backslash\mathcal{Q}^\tau \textit{ we find } (\tilde{\lambda}, \varphi)_{L^2(\mathcal{Q}^\tau)} = 0 \tag{7.23g}$$

for $[p_\varepsilon, \xi_\varepsilon]_{L^2(S)} := \lim_{\gamma \to \infty} (p_{\varepsilon_\gamma}, \xi_{\varepsilon_\gamma})_{L^2(S)}$.

Proof. Let a vanishing sequence $\{\varepsilon\}$ be given. First we analyze the behavior of the primal variables. Let $(y_\varepsilon, u_\varepsilon, \xi_\varepsilon)$ satisfy the complementarity system (7.20c), (7.20d) for any ε. As already discussed, this is equivalent to y_ε solving (VI_ε^T) for the data $f + u_\varepsilon$. Due to the regularity result (7.8), the sequence of slack variables

$$\xi_\varepsilon = D_t y_\varepsilon + A^\varepsilon y_\varepsilon - f - u_\varepsilon$$

is bounded in $\mathfrak{H}$ (see Remark 7.3.1). By assumption y_ε is bounded in $\mathfrak{V}$ and thus, as a consequence of Lemma 7.2.4, $D_t y_\varepsilon$ is bounded in $\mathfrak{H}$ and so y_ε is bounded in $\hat{W}(0,T)$ and $W(0,T)$ by the continuous embedding. According to the established bounds we can find a subsequence again denoted by ε such that

$$y_\varepsilon \rightharpoonup \tilde{y} \text{ in } \hat{W}(0,T),\ \ u_\varepsilon \rightharpoonup \tilde{u} \text{ in } \mathfrak{H},\ \ \xi_\varepsilon \rightharpoonup \tilde{\xi} \text{ in } \mathfrak{H}$$

Similar to Lemma 7.2.5 we find

$$D_t \tilde{y} + A^0 \tilde{y} - f - \tilde{u} = \tilde{\xi},\ \tilde{y}(0) = u_0,\ \tilde{y} - \psi \geq 0,\ \tilde{\xi} \geq 0, (\tilde{y} - \psi, \tilde{\xi}) = 0$$

for the weak limits. Since the boundary condition $y(x,t) = 0$ on $\partial\Omega \times [0,T]$ is incorporated in $\mathfrak{V}$, it is met as well and the complementarity system (7.23c) to (7.23e) is established.
Next we discuss the adjoint state. By (7.20b) and the assumed boundedness of u_ε we find $\{p_\varepsilon\}$ to be bounded in $\mathfrak{H}$ converging, along a further subsequence, weakly to some limit element

$$p_\varepsilon \rightharpoonup \tilde{p} = \beta \tilde{u}$$

providing (7.23b). Considering the adjoint equation (7.20a) and using test functions φ from the closed linear subspace $\hat{W}_0(0,T) \subset \hat{W}(0,T) \subset W(0,T)$ we obtain

$$(y_\varepsilon - y^d, \varphi) - \tilde{\beta}\langle \Delta y_\varepsilon, \varphi\rangle - \langle \lambda_\varepsilon, \varphi\rangle_{W^*,W} + \langle p_\varepsilon, (D_t + A^\varepsilon)\varphi\rangle = 0.$$

Separating the second order part $\varepsilon \Delta p_\varepsilon$ from the adjoint operator we find, since $\hat{W}_0(0,T) \subset \mathfrak{V}$, that

$$\langle \lambda_\varepsilon + \varepsilon \Delta p_\varepsilon, \varphi\rangle_{\hat{W}_0(0,T)^*, \hat{W}_0(0,T)} = (y_\varepsilon - y^d, \varphi) - \tilde{\beta}\langle \Delta y_\varepsilon, \varphi\rangle + (p_\varepsilon, (D_t + A^0)\varphi).$$

By the regularity of φ and the established bounds on y_ε and p_ε, $\lambda_\varepsilon + \varepsilon\Delta p_\varepsilon$ is bounded in $\hat{W}_0(0,T)^*$, a closed linear subspace of the Hilbert space $\hat{W}(0,T)$, and thus, along a subsequence denoted by ε, we get

$$\lambda_\varepsilon + \varepsilon\Delta p_\varepsilon \rightharpoonup \tilde{\lambda} \text{ in } \hat{W}_0^*(0,T).$$

For $\varepsilon \to 0$ we obtain

$$(\tilde{y} - y^d, \varphi) - \tilde{\beta}\langle\Delta\tilde{y}, \varphi\rangle + (\tilde{p}, (D_t + A^0)\varphi) - \langle\tilde{\lambda}, \varphi\rangle_{\hat{W}_0(0,T)^*, \hat{W}_0(0,T)} = 0$$

and have derived (7.23a).
Since $\tilde{\xi} \in \mathfrak{H}$ admits a pointwise representation, the set $S = \{x \in \mathcal{Q} : \tilde{\xi}(x) > 0\}$ is well defined. From the proof of Theorem 7.4.2 we know, that $\lim_{\gamma\to\infty}(p_{\varepsilon_\gamma}, \xi_{\varepsilon_\gamma})_{L^2(\omega)} = 0$ for all subsets $\omega \subset \mathcal{Q}$ and $\varepsilon > 0$ which in particular holds true for $\omega = S$. Considering the limit $\varepsilon \to 0$, we find, that (7.20g) has to hold.
To prove the asserted behavior of $\tilde{\lambda}$ on $\mathcal{Q}_+ = \{x \in \mathcal{Q} : \tilde{y}(x) > 0\}$ we introduce the following set, defined for an arbitrary $s \in \mathbb{R}$, $s > 0$.

$$\mathcal{Q}_+^s := \{x \in \mathcal{Q} : \tilde{y}(x) \geq \psi(x) + s\}.$$

Obviously, we have $\mathcal{Q}_+ = \bigcup_s \mathcal{Q}_+^s$. Given any $\tau > 0$ we find, analogously to Theorem 6.2.4, some $s_\tau > 0$ with

$$|\mathcal{Q}_+ \backslash \mathcal{Q}_+^{s_\tau}| \leq \tau/3. \tag{7.24}$$

By the continuous embedding $\hat{W}(0,T) \to W(0,T)$, a weakly convergent and thus bounded subsequence (see Theorem A.4.3) in $\hat{W}(0,T)$ contains a weakly convergent subsequence in $W(0,T)$ as the boundedness also holds in the latter space. By the compact embedding into $\mathfrak{H}$ we find a further subsequence such that $y_\varepsilon \to \tilde{y}$ in $\mathfrak{H}$ and according to Theorem A.4.9 there exist another subsequence denoted by ε such that $y_\varepsilon(x) \to y(x)$ almost everywhere in $\mathcal{Q}$. The Theorem of Egorov A.4.10 yields for any $\tilde{\tau} > 0$ the existence of a measurable set $\mathcal{Q}_{\tilde{\tau}}$ with $|\mathcal{Q}\backslash\mathcal{Q}_{\tilde{\tau}}| \leq \tilde{\tau}$ and $y_\varepsilon \to y^*$ uniformly almost everywhere on $\mathcal{Q}_{\tilde{\tau}}$. Considering (7.24) and setting $\tilde{\tau} = \tau/3$ we find for $\tilde{\mathcal{Q}}^\tau = (\mathcal{Q}_+^{s_\tau} \cap \mathcal{Q}_{\tau/3})$

$$|\mathcal{Q}_+ \backslash \tilde{\mathcal{Q}}^\tau| = |\mathcal{Q}_+ \backslash \mathcal{Q}_+^{s_\tau} \cup \mathcal{Q}_+ \backslash \mathcal{Q}_{\tau/3}| \leq \tau/3 + \tau/3 = (2/3)\tau.$$

The uniform convergence of y_ε on $\tilde{\mathcal{Q}}^\tau$ implies the existence of some $\tilde{\varepsilon} > 0$ such that

$$y_\varepsilon(x) > \psi(x) \text{ for all } x \in \tilde{\mathcal{Q}}^\tau \text{ and } \varepsilon > \tilde{\varepsilon}$$

implying $\tilde{\mathcal{Q}}^\tau \subset \{x \in \mathcal{Q} : y_\varepsilon(x) > \psi(x)\}$ for all $\varepsilon > \tilde{\varepsilon}$. Next we consider a sequence of positive real numbers τ_ε with $\sum_{\varepsilon \geq \tilde{\varepsilon}} \tau_\varepsilon = \tau/3$. Since $(y_\varepsilon, u_\varepsilon, \xi_\varepsilon, p_\varepsilon, \lambda_\varepsilon)$ satisfies (7.20) and (7.21) for every $\varepsilon > 0$ we find for each of the τ_ε a set E_{τ_ε} with $|\{y_\varepsilon > \psi\}\backslash E_{\tau_\varepsilon}| \leq \tau_\varepsilon$ and

$$\lim_{\gamma\to\infty}(\lambda_{\varepsilon_\gamma} - \nu_{\varepsilon_\gamma}\xi_{\varepsilon_\gamma}, \varphi) = (\lambda_\varepsilon, \varphi) = 0$$

for all $\varphi \in C_c^\infty(\mathcal{Q})$ with $\varphi(x) = 0$ for all $x \in \mathcal{Q}\backslash E_{\tau_\varepsilon}$ which stems from the natural inclusion $C_c^\infty(\mathcal{Q}) \subset W(0,T)$. The monotonicity of the Lebesgue measure yields

$$|\tilde{\mathcal{Q}}^\tau \backslash E_{\tau_\varepsilon}| \leq |\{y_\varepsilon > 0\}| \backslash E_{\tau_\varepsilon}$$

for all $\varepsilon \geq \tilde{\varepsilon}$. Now we set $\mathcal{Q}^\tau = \tilde{\mathcal{Q}}^\tau \cap \bigcap_{\varepsilon \geq \tilde{\varepsilon}} E_{\tau_\varepsilon}$ and obtain

$$\lim_{\varepsilon \to 0} (\lambda_\varepsilon + \varepsilon \Delta p_\varepsilon, \varphi) = (\lambda^*, \varphi) = 0$$

for all $\varphi \in C_c^\infty(\mathcal{Q})$ with $\varphi(x) = 0$ for all $x \in \mathcal{Q} \backslash \mathcal{Q}^\tau$ for all $\varepsilon \geq \tilde{\varepsilon}$. Analogously to Theorem 6.2.4 we further obtain

$$|\mathcal{Q}_+ \backslash \mathcal{Q}^\tau| \leq (2/3)\tau + \sum_{\varepsilon \geq \tilde{\varepsilon}} \tau_\varepsilon = \tau.$$

By the inclusion $C_c^\infty(\mathcal{Q}) \subset \hat{W}(0,T)$ this finishes the proof. □

Again we point out that this is, similar to the non parametric case discussed in Chapter 6, a constructive first order necessary optimality condition. The consistency proof showed, that any solution to the original problem $(\tilde{P}^T)$ is approximated by solutions to the problems $(\tilde{P}_\varepsilon^T)$. For these approximating solutions a first order necessary optimality condition was derived which under suitable conditions converge to system (7.23). Since the conditions needed for the convergence process are met by a sequence of minimizers to the problems $(\tilde{P}_\varepsilon^T)$, the solutions of the original problem have to satisfy this system. Therefore, we have a necessary first order optimality system at hand.

Remark 7.4.2. *Similar to Remark 6.2.1, it is an open question whether Problem* $(\tilde{P}^T)$ *can be reduced to the case, where all admissible control functions admit a representation of the form* (7.12). *If this possible, the original problem would reduce to*

$$\begin{array}{ll} \min & \frac{1}{2}|w - y^d|^2 + \frac{\tilde{\beta}}{2}\|w\|^2 + \frac{\beta}{2}|D_t w + A^0 w| \\ s.t. & w \in \hat{W}(0,T) \\ & w \in \mathcal{K} \\ & w(0) = u_0, \end{array}$$

a problem, that is substantially simpler than the original posed one.

7.5 Algorithmic Treatment

Without loss of generality we restrict ourselves to $\psi \equiv 0$.

7.5.1 The Algorithm

The constructive nature of the proof of Theorem 7.4.3 suggests an algorithm for the computation of elements satisfying (7.23). The resulting procedure is presented in Algorithm 5. It is based on the continuation technique derived in [95] since, before starting the refinement process for ε, we have to find points satisfying (7.20). Therefore, the updating of the parameter α, κ is related to the penalization parameter γ according to Assumption 7.1. Our update rule is already incorporated in the algorithm.

Algorithm 5 Outer Loop

DATA: $y_d, f, \bar{\lambda}, (\gamma, \alpha, \kappa) > 0, \beta_\gamma > 1, \varepsilon > 0, \beta_\varepsilon \in (0,1),\ \beta_\gamma > 1$
Initialize (y^0, u^0, ξ^0, ν^0)
Repeat
  Repeat
    - Compute Stationary Point $(y^{k+1}, u^{k+1}, \xi^{k+1}, \nu^{k+1})$ of $(P^T_{\varepsilon_{\gamma,\alpha,\kappa}})$ with Initial Value (y^k, u^k, ξ^k, ν^k) using a Semismooth Newton Method and nested grids
    - Update $\gamma^+ = \gamma\beta_\gamma$
    - Compute $\kappa(\gamma^+) = \kappa_0(\gamma^+)^{-0.55}, \alpha(\gamma^+) = \alpha_0(\gamma^+)^{-1/2}$
  Until Refinement Criterion is met
Update $\varepsilon^+ = \varepsilon\beta_\varepsilon$
Until Stopping Criterion is met

Reduction and Discretization of the Stationarity System

For an efficient numerical treatment of (7.14) we have to reduce the system. First we reformulate the complementarity conditions (7.14e) and (7.14f) as in Section 6.3 using the pointwise max-operator. For arbitrary positive constants c_μ, c_ν we find the conditions to be equivalent to

$$\begin{aligned} \xi - \max\{0, \xi - c_\mu \mu\} &= 0 \\ \nu - \max\{0, \nu + c_\nu((\xi, y) - \alpha)\} &= 0 \end{aligned} \tag{7.25}$$

Using (7.14d) and (7.14c) we obtain expressions for μ and p providing

$$\xi = \kappa^{-1}(\beta u - \nu y)^+$$

for $c_\mu = \kappa^{-1}$. Since in each step of the procedure $\varepsilon > 0$ holds, the adjoint state admits additional regularity $u \in \mathfrak{V}$. For $v \in H^1_0(\Omega)$ $v^+ \in H^1_0(\Omega)$ holds (see [94]) and consequently $\xi \in \mathfrak{V}$ is satisfied at all stages of the algorithm. Considering (7.25) we define $c_\nu = \kappa$ and define

$$\chi_\alpha = \begin{cases} 1 & \text{if} \quad \nu - c_\nu(\alpha - (\xi, y)) > 0 \quad \Leftrightarrow \nu - \kappa\alpha + ((\beta u - \nu y)^+, y) > 0 \\ 0 & \text{else} \end{cases}$$

obtaining the expression

$$0 = \nu - \chi_\alpha \nu - \chi_\alpha((\beta u - \nu y)^+, y) + \chi_\alpha \kappa\alpha$$

for (7.25). Introducing the indicator functions $\chi_y = \{\bar{\lambda} - \gamma y > 0\}$ and $\chi_u = \{(\beta u - \nu y) > 0\}$, dividing the first equation by β and defining $\tilde{y}^d = y^d/\beta$ we obtain the reduced system

$$\begin{aligned} 0 &= -D_t u + (A^\varepsilon)^* u + \frac{\nu}{\kappa}\chi_u u - \frac{\tilde{\beta}}{\beta}\Delta y + (\frac{1}{\beta} I + \frac{\gamma}{\beta}\chi_y - \frac{\nu^2}{\beta\kappa}\chi_u) y - \tilde{y}^d - \frac{1}{\beta}\chi_y \bar{\lambda} \\ 0 &= D_t y + A^\varepsilon y + \frac{\nu}{\kappa}\chi_u y - (I + \frac{\beta}{\kappa}\chi_u) u - f \\ 0 &= \nu - \chi_\alpha \nu - \beta\chi_\alpha(\chi_u u, y) + \nu\chi_\alpha(\chi_u y, y) + \chi_\alpha \kappa\alpha \end{aligned}$$

with initial/terminal conditions

$$y(\cdot,0)=u_0 \quad \text{and} \quad u(\cdot,T)=0$$

The nonlinear mapping

$$F=(F_1,F_2,F_3):W(0,T)\times W(0,T)\times\mathbb{R}\to\mathfrak{V}^*\times\mathfrak{V}^*\times\mathbb{R}$$

defined

$$F_1(y,u,\nu)=-D_tu+(A^\varepsilon)^*u+\frac{\nu}{\kappa}\chi_u u-\frac{\tilde{\beta}}{\beta}\Delta y+(\frac{1}{\beta}I+\frac{\gamma}{\beta}\chi_y-\frac{\nu^2}{\beta\kappa}\chi_u)y-\tilde{y}^d-\frac{1}{\beta}\chi_y\bar{\lambda} \tag{7.26a}$$

$$F_2(y,u,\nu)=D_ty+A^\varepsilon y+\frac{\nu}{\kappa}\chi_u y-(I+\frac{\beta}{\kappa}\chi_u)u-f \tag{7.26b}$$

$$F_3(y,u,\nu)=\nu-\chi_\alpha\nu-\beta\chi_\alpha(\chi_u u,y)+\nu\chi_\alpha(\chi_u y,y)+\chi_\alpha\kappa\alpha \tag{7.26c}$$

has the following property.

Lemma 7.5.1. *F is Newton differentiable in the sense of Definition 2.4.1 provided $u\in W(0,T)$ is satisfied.*

Proof. Since $W(0,T)\to L^{2+\theta}(\mathcal{Q})$ for some $\theta>0$ sufficiently small (see Lemma A2.1), Newton differentiability of F follows from the same arguments as in Lemma 6.3.1 if $u\in W(0,T)$ is ensured. □

Utilizing the preceding result, a root of F can be found by the semismooth Newton method. With the notation

$$(y_+,u_+,\nu_+)=(y,u,\nu)+(\delta y,\delta u,\delta\nu)$$

the system which has to be solved can be reformulated as

$$\begin{aligned}(-\frac{\tilde{\beta}}{\beta}\Delta\cdot+\frac{1}{\beta}I+\frac{\gamma}{\beta}\chi_y-\frac{\nu^2}{\beta}\chi_u)y_++(-D_t+(A^\varepsilon)^*+\frac{\nu}{\kappa}\chi_u)u_++\frac{1}{\kappa}(\chi_u(u-\frac{2\nu}{\beta}y))\nu_+&\\ =\tilde{y}^d+\frac{1}{\beta}\chi_y\bar{\lambda}+\frac{1}{\kappa}(\chi_u(u-\frac{2\nu}{\beta}y))\nu&\qquad(7.27\text{a})\\ (D_t+A^\varepsilon+\frac{\nu}{\kappa}\chi_u)y_+-(I+\frac{\beta}{\kappa}\chi_u)u_++\frac{1}{\kappa}\chi_u y\nu_+=f+\frac{1}{\kappa}\chi_u y\nu&\qquad(7.27\text{b})\\ -\beta\chi_\alpha(\chi_u u,y_+)+2\nu\chi_\alpha(\chi_u y,y_+)-\beta\chi_\alpha(\chi_u y,u_+)+(1-\chi_\alpha(1-(\chi_u y,y)))\nu_+&\\ =2\nu\chi_\alpha(\chi_u y,y)-\beta\chi_\alpha(\chi_u y,u)-\chi_\alpha\kappa\alpha&\qquad(7.27\text{c})\end{aligned}$$

Since u_+ solves a parabolic partial differential equation with data in $\mathfrak{V}^*$, it preserves the regularity of $u\in W(0,T)$ according to Theorem 2.2.5.
(7.27) has still to be considered in the corresponding function spaces. For a numerical realization we have to discretize the system. We restrict ourselves to a model domain $\Omega=(0,1)\times(0,1)$ for the time interval $(0,T)$. Ω is discretized with a uniform mesh of width $h=1/(N+1)$ where $N\in\mathbb{N}$ is the number of inner grid points of the interval $[0,1]$. The time interval is discretized by a uniform mesh of width $\Delta t=T/m$ with $m\in\mathbb{N}$ representing the number of time steps. Thus the discretized domain is defined by the points

$$\mathcal{Q}_h=\{(x_{i,j},t_k)=(ih,jh,k\Delta t):1\le i,j\le N,0\le k\le m\}$$

and the piecewise linear grid functions are defined by vectors $\vec{y}, \vec{u} \in \mathbb{R}^{N^2(m+1)}$. For notational simplicity, we utilize a superscript to indicate the time step, i.e.

$$\mathbf{v}^{(k)} = (\vec{v}_{kN^2+1}, \ldots, \vec{v}_{(k+1)N^2})^\top$$

represents the values of the approximating piecewise linear function on the inner nodes of the computational domain at time level $k \in \{0, ..., m\}$. Moreover, we separate vectors $\vec{v} \in \mathbb{R}^{N^2(m+1)}$ in the following way.

$$\vec{v} = (\mathbf{v}^{(0)}, \mathbf{v}, \mathbf{v}^{(m)})^\top \text{ with } \mathbf{v} \in \mathbb{R}^{(m-1)N^2}$$

For a fixed time step the discretization of the differential operator will be done by $P1$ finite elements (recall Definition 6.3.1 from the preceding Chapter). For the parabolic problem we will utilize mass lumping in form of row sum lumping (see [66]) and thus M as well as M_u, M_y are diagonal matrices where the latter ones contain strictly positive entries on the main diagonal depending on whether the corresponding node is active or inactive with respect to $\bar{\lambda}^{(k)} - \gamma y^{(k)}$ or $\beta u^{(k)} - \nu y^{(k)}$. The time derivative in the adjoint equation (7.27a) is discretized by a forward finite difference providing in time step k

$$D_t u^{(k)} = \Delta t^{-1}(u^{(k+1)} - u^{(k)})$$

for $0 \leq k \leq m-1$. In the primal equation (7.27a) we use a backward finite difference

$$D_t y^{(k)} = \Delta t^{-1}(y^{(k)} - y^{(k-1)})$$

for $1 \leq k \leq m$. According to the signs of the derivatives w.r.t. time in (7.27b) and (7.27a) these choices yield implicit formulations for the time stepping in the primal and adjoint equation. The integration w.r.t. time in the inner product of $L^2(\mathcal{Q})$ is approximated by the trapezoidal rule. Thus we obtain

$$(v, w)_{L^2(\mathcal{Q})} = \Delta t \left(\frac{1}{2}((v^{(0)}, w^{(0)})_{L^2(\Omega)} + (v^{(m)}, w^{(m)})_{L^2(\Omega)}) + \sum_{k=1}^{m-1} (v^{(k)}, w^{(k)})_{L^2(\Omega)} \right). \quad (7.28)$$

For a discretization of (7.27) we first define building blocks by the following objects.

$$\begin{aligned}
M_t &= \frac{1}{\Delta t} M & S_k &= \frac{\tilde{\beta}}{\beta} S + \frac{1}{\beta} M + \frac{\gamma}{\beta} M_{y^{(k)}} - \frac{\nu^2}{\beta\kappa} M_{u^{(k)}} \\
B_k &= M_t + A^\varepsilon + \frac{\nu}{\kappa} M_{u^{(k)}} & B_k^* &= M_t + (A^\varepsilon)^* + \frac{\nu}{\kappa} M_{u^{(k)}} \\
M_k &= -M - \frac{\beta}{\kappa} M_{u^{(k)}} & \mathbf{M} &= blkdiag(M, \ldots, M) \\
M_{\mathbf{u}} &= blkdiag(M_{\mathbf{u}^{(1)}}, \ldots, M_{\mathbf{u}^{(m-1)}}) & C_k &= \frac{1}{\kappa} M_{u^{(k)}} \mathbf{y}^{(k)} \\
D_k &= M_{u^{(k)}} \left(\frac{1}{\kappa} u^{(k)} - \frac{2\nu}{\beta\kappa} y^{(k)} \right)
\end{aligned}$$

Here the first 5 matrices are in $\mathbb{R}^{N^2 \times N^2}$, the block diagonal matrices $\mathbf{M}, M_{\mathbf{u}}$ are elements of $\mathbb{R}^{(m-1)N^2 \times (m-1)N^2}$, $\vec{M}_{\mathbf{u}} \in \mathbb{R}^{(m+1)N^2 \times (m+1)N^2}$ and the vectors C_k, D_k are in $\mathbb{R}^{N^2}$.

Now the discretized version of (7.27) is given as

$$\mathfrak{S}\mathbf{y}_+ + \mathfrak{B}^*\mathbf{u}_+ + \mathfrak{D}\nu_+ = \mathbf{M}\tilde{\mathbf{y}}^d + \frac{1}{\beta}M_{\mathbf{y}}\bar{\lambda} + M_{\mathbf{u}}(\frac{1}{\kappa}\mathbf{u} - \frac{2\nu}{\kappa\beta}\mathbf{y})\nu$$

$$\mathfrak{B}\mathbf{y}_+ + \mathfrak{M}\mathbf{u}_+ + \mathfrak{C}\nu_+ = \mathbf{M}\mathbf{f} + \frac{1}{\kappa}M_u\mathbf{y}\nu + \underbrace{(M_t u_0, 0, \dots, 0)^\top}_{\mathfrak{K}}$$

$$-M_t\mathbf{y}_+^{(m-1)} + B_m\mathbf{y}_+^{(m)} + C_m\nu_+ = Mf^{(m)} + \frac{1}{\kappa}M_{\mathbf{u}^{(m)}}\mathbf{y}^{(m)}\nu$$

$$B_0^*\mathbf{u}_+^{(0)} - M_t\mathbf{u}_+^{(1)} + D_0\nu_+ = M(\tilde{y}^d)^{(0)} + M_{y^{(0)}}\frac{1}{\beta}(\bar{\lambda})^{(0)} + M_{\mathbf{u}^{(0)}}(\frac{1}{\kappa}u^{(0)} - \frac{2\nu}{\kappa\beta}u_0)\nu - S_0u_0$$

and

$$\Delta t\chi_\alpha(2\nu\mathbf{y} - \beta\mathbf{u})^\top M_u\mathbf{y}_+ + \Delta t\chi_\alpha\nu(\mathbf{y}^{(m)})^\top M_{u^{(m)}}\mathbf{y}_+^{(m)} - \beta\Delta t(\chi_\alpha/2)u_0^\top M_{u^{(0)}}\mathbf{u}_+^{(0)} -$$

$$\Delta t\chi_\alpha\beta\mathbf{y}^\top M_u\mathbf{u}_+ + (1 - \chi_\alpha(1 - \Delta t(\frac{1}{2}(u_0^\top M_{u^{(0)}}u_0 + (\mathbf{y}^{(m)})^\top M_{u^{(m)}}\mathbf{y}^{(m)}) + \mathbf{y}^\top M_u\mathbf{y})))\nu_+$$

$$= \chi_\alpha\Delta t(\frac{1}{2}((2\nu u_0 - \beta u^{(0)})^\top M_{u^{(0)}}u_0 + 2\nu(y^{(m)})^\top M_{u^{(m)}}y^{(m)}) + (2\nu\mathbf{y} - \beta\mathbf{u})^\top M_{\mathbf{u}}\mathbf{y}) - \chi_\alpha\kappa\alpha$$

for the following matrices in $\mathbb{R}^{(m-1)N^2\times(m-1)N^2}$ and vectors in $\mathbb{R}^{(m-1)N^2}$.

$$\mathfrak{S} = \begin{pmatrix} S_1 & & \\ & \ddots & \\ & & S_{m-1} \end{pmatrix}, \qquad \mathfrak{M} = \begin{pmatrix} M_1 & & \\ & \ddots & \\ & & M_{m-1} \end{pmatrix},$$

$$\mathfrak{B}* = \begin{pmatrix} B_1^* & -M_t & & \\ & \ddots & \ddots & \\ & & \ddots & -M_t \\ & & & B_{m-1}^* \end{pmatrix}, \qquad \mathfrak{B} = \begin{pmatrix} B_1 & & & \\ -M_t & \ddots & & \\ & \ddots & \ddots & \\ & & -M_t & B_{m-1} \end{pmatrix}$$

$$\mathfrak{D} = (D_{(1)}, \dots, D_{(m-1)})^\top, \qquad \mathfrak{C} = (C_{(1)}, \dots, C_{(m-1)})^\top$$

$\mathfrak{M}$ is a diagonal matrix with nonzero diagonal elements and thus invertible. Consequently, the second line can be used to substitute

$$\mathbf{u}_+ = \mathfrak{M}^{-1}\mathbf{M}\mathbf{f} + \frac{1}{\kappa}\mathfrak{M}^{-1}M_{\mathbf{u}}\mathbf{y}\nu + \mathfrak{M}^{-1}\mathfrak{K} - \mathfrak{M}^{-1}\mathfrak{B}\mathbf{y}_+ - \mathfrak{M}^{-1}\mathfrak{C}\nu_+$$

providing the expression

$$u_+^{(1)} = M_1^{-1}Mf^{(1)} + \frac{\nu}{\kappa}M_1^{-1}M_{u^{(1)}}y^{(1)} + M_1^{-1}M_tu_0 - M_1^{-1}B_1y_+^{(1)} - \frac{1}{\kappa}M_1^{-1}M_{u^{(1)}}y^{(1)}\nu_+.$$

Utilizing these identities the system can further be reduced to

$$
\begin{aligned}
&(\mathfrak{S} - \mathfrak{M}^{-1}\mathfrak{B}^*\mathfrak{B})\mathbf{y}_+ + (\mathfrak{D} - \mathfrak{M}^{-1}\mathfrak{B}^*\mathfrak{C})\nu_+ \\
&= \mathbf{M}(\tilde{\mathbf{y}}^d - \mathfrak{M}^{-1}\mathfrak{B}^*\mathbf{f}) + \frac{1}{\beta}M_{\mathbf{u}}\bar{\lambda} - \mathfrak{M}^{-1}\mathfrak{B}^*\mathfrak{K} + \frac{\nu}{\kappa}M_{\mathbf{u}}(\mathbf{u} - (\frac{2\nu}{\beta}I + \mathfrak{M}^{-1}\mathfrak{B}^*)\mathbf{y}) \\
&- M_t\mathbf{y}_+^{(m-1)} + B_m\mathbf{y}_+^{(m)} + C_m\nu_+ \\
&= Mf^{(m)} + \frac{\nu}{\kappa}M_{\mathbf{u}^{(m)}}\mathbf{y}^{(m)} \\
&B_0^*\mathbf{u}_+^{(0)} + M_1^{-1}M_tB_1\mathbf{y}_+^{(1)} + (D_0 + M_1^{-1}M_tC_1)\nu_+ \\
&= M((\tilde{\mathbf{y}}^d)^{(0)} + \frac{1}{\beta}M_{y^{(0)}}\bar{\lambda}^{(0)} + M_tM_1^{-1}Mf^{(1)}) + M_{\mathbf{u}^{(\mathbf{0})}}(\frac{1}{\kappa}\mathbf{u}^{(0)} - \frac{2\nu}{\kappa\beta}u_0)\nu - S_0u_0 \\
&\quad + \frac{\nu}{\kappa}M_{\mathbf{u}^{(1)}}M_1^{-1}M_t\mathbf{y}^{(1)} + M_t^2M_1^{-1}u_0 \\
&\chi_\alpha\Delta t((\mathbf{y}^\top(2\nu I + \beta(\mathfrak{M}^{-1}\mathfrak{B})) - \beta\mathbf{u}^\top)M_{\mathbf{u}}\mathbf{y}_+ + \nu\mathbf{y}^{(m)}M_{\mathbf{u}^{(m)}}\mathbf{y}_+^{(m)} - \frac{\beta}{2}u_0^\top M_{\mathbf{u}^{(0)}}\mathbf{u}_+^{(0)})+ \\
&(1 - \chi_\alpha + \chi_\alpha\Delta t\frac{1}{2}(u_0^\top M_{\mathbf{u}^{(0)}}u_0 + (\mathbf{y}^{(m)})^\top M_{\mathbf{u}^{(m)}}\mathbf{y}^{(m)}) + \chi_\alpha\Delta t\mathbf{y}^\top M_u(\mathbf{y} + \beta\mathfrak{M}^{-1}\mathfrak{C}))\nu_+ \\
&= \chi_\alpha\Delta t(\frac{1}{2}((2\nu u_0 - \beta u^{(0)})^\top M_{u^{(0)}}u_0 + 2\nu(y^{(m)})^\top M_{u^{(m)}}y^{(m)}) + (2\nu\mathbf{y} - \beta\mathbf{u})^\top M_{\mathbf{u}}\mathbf{y}) - \chi_\alpha\kappa\alpha \\
&\quad + \Delta t\chi_\alpha\beta\mathbf{y}^\top M_{\mathbf{u}}\mathfrak{M}^{-1}(\mathbf{Mf} + \frac{\nu}{\kappa}M_{\mathbf{u}}\mathbf{y} + \mathfrak{K})
\end{aligned}
$$

The reduced system matrix is given as

$$
\begin{pmatrix}
 & & \mathfrak{S} & -\mathfrak{M}^{-1}\mathfrak{B}^*\mathfrak{B} & & & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \begin{matrix}0\\ \vdots\\ 0\end{matrix} & \mathfrak{D} - \mathfrak{M}^{-1}\mathfrak{B}^*\mathfrak{C} \\
0 & & & \dots & 0 & -M_t & B_m & 0 & C_m \\
M_1^{-1}M_tB_1 & & 0 & 0 & & \dots & 0 & B_0^* & D_0 + M_1^{-1}M_tC_1 \\
\chi_\alpha\Delta t((\mathbf{y}^\top(2\nu I+ & \beta(\mathfrak{M}^{-1}\mathfrak{B})) & & -\beta\mathbf{u}^\top)M_{\mathbf{u}} & & & L_1 & L_2 & L_3
\end{pmatrix}
\tag{7.29}
$$

with

$$
\begin{aligned}
L_1 &= \chi_\alpha\Delta t\nu\mathbf{y}^{(m)}M_{\mathbf{u}^{(m)}}, \quad L_2 = -\chi_\alpha\Delta t\frac{\beta}{2}u_0^\top M_{\mathbf{u}^{(0)}} \\
L_3 &= (1 - \chi_\alpha + \chi_\alpha\Delta t\frac{1}{2}(u_0^\top M_{\mathbf{u}^{(0)}}u_0 + (\mathbf{y}^{(m)})^\top M_{\mathbf{u}^{(m)}}\mathbf{y}^{(m)}) + \chi_\alpha\Delta t\mathbf{y}^\top M_u(\mathbf{y} + \beta\mathfrak{M}^{-1}\mathfrak{C}))
\end{aligned}
$$

$\mathfrak{S} - \mathfrak{M}^{-1}\mathfrak{B}^*\mathfrak{B}$ is a block diagonal matrix with $S_k - M_k^{-1}(B_k^*B_k + M_t^2)$ on the main diagonal for $k = 1, \dots, m-1$, $-M_{-1}^{-1}M_tB_k$ on the first upper off-diagonal and $-M_k^{-1}M_tB_k^*$ on the first lower off-diagonal for $k = 2, \dots, m-1$. Thus the final number of unknowns is $(m+1)N^2 + 1$. In case of $\chi_\alpha = 0$ the system reduces once more since the last line provides $\nu_+ = 0$ and only the $((m+1)N^2 + 1)$th minor needs to be considered.

Note that the computation of $\mathbf{y}^{(m)}$ and $\mathbf{u}^{(0)}$ is only necessary for the evaluation of (7.28). Leaving out $\frac{1}{2}((v^{(0)}, w^{(0)})_{L^2(\Omega)} + (v^{(m)}, w^{(m)})_{L^2(\Omega)})$ reduces the system again (number of unknowns is $(m-1)N^2 + 1$) but introduces a further discretization error.

Parameter, Stopping Criteria and used Software

The problem will be discretized on a hierarchy of meshes of width $2^{-(4+i)}$ for $i = 1, 2$. As in Chapter 6, passing to the next finer grid will be performed whenever γ exceeds a threshold depending on the discretization parameter h. We utilized the heuristic developed in [95] which depends on approximation theory for the Tikhonov term (see e.g. [51]) and estimates for the discretization error for parabolic PDE's. It provides a balancing of approximation and discretization errors and yields the following update rule. The mesh has to be refined if

$$\gamma \geq \frac{c_{Grid}}{((h^2) + \Delta t)^2}.$$

Whenever the refinement process takes place, the variables have to be passed to the finer grid. In the implementation we decided for linear interpolation via the Matlab built-in function *interp3* and refine the mesh simultaneously with respect to the spatial variables and time. The inner loop terminates, if γ exceeds the threshold for the finest grid we consider. We have decided to choose $c_{Grid} = 4$.

ε is decreased by multiplying with some fixed constant $\beta_\varepsilon \in (0, 1)$ and terminates, if $\varepsilon \leq 10^{-7}$.

The sparse linear system, occuring in any step of the Semismooth Newton method, is solved by Matlab built-in functions as well. Large linear systems with sparse coefficient matrices can efficiently be solved by iterative solvers like *GMRES* or *BICGStab*. We decided for the latter one since it performed better in the test runs. Thus each system is solved by the function *bicgstab* with maximal number of iterations equal to the number of unknowns and tolerance 10^{-11}. The fix point procedure terminates if the relative residual of the iterate is smaller than the predefined tolerance.

In [95] a preconditioner was suggested which performed well on the problems considered there. The basic idea is to consider the discretized operators without terms depending on the previous iterates (in our setting $M_{\mathbf{u}}$ and $M_{\mathbf{y}}$) and to perform an incomplete LU factorization of the corresponding $((m+1)N^2 + 1)$th minor in (7.29). Then they use

$$\tilde{L} = \begin{pmatrix} L & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{U} = \begin{pmatrix} U & 0 \\ 0 & 1 \end{pmatrix}$$

as preconditioner. The reported gain was, that the preconditioner construction, although performed only once per mesh, provided a significant decrease in the iteration numbers of the fixpoint procedure *bicgstab*.

We did test runs using this approach with two possible extensions to our setting. Recall that here the operator changes in the outer loop. First we tried the same as in [95] while knowing, that when the outer loop starts, we have to perform one incomplete LU factorization per step when ε is decreased. Second we used an incomplete LU factorization of the discretized differential operator with $\varepsilon = TOL_\varepsilon$ from the beginning. For both methods we observed a decrease in the number of iterations for most of the γ updates ($\approx \frac{1}{2}$ of the original iterations). The second choice always performed better than the first.

Unfortunately, this strategy did not pay off in computation time since for both possibilities, the fixpoint procedure needed more time to solve the system when using the preconditioners than without.

We initialized the algorithm with the feasible point

$$y^{(0)}(x,t) = u_0(x), \;\; u^{(0)}(x,t) = -f(x,t) + A^{\varepsilon}u_0(x,t), \nu^{(0)} = 0$$

and used a globalization as in Algorithm 4, Chapter 6 with the same parameters.

The numerical computations were performed on the computer cluster of the Humboldt-Universität zu Berlin with Dual Xeon Quad Core nodes and 48 GB memory.

7.5.2 Examples

We consider the optimization problem

$$\begin{array}{lrl} \min & \frac{1}{2}|y-y^d|^2 + \frac{\tilde{\beta}}{2}\|y\|^2 + \frac{\beta}{2}|u| & \\ \text{s.t.} & y \in \mathcal{K}, \;\; \langle D_t y + \mathbf{b}\cdot\nabla y + b^0 y - f - u, v - y\rangle & \geq 0 \quad \forall v \in \mathcal{K} \\ & y(0) & = 0 \end{array}$$

for $\mathbf{b} = (0.2; -0.5)^\top$, $b^0 = 0.1$ and $\mathcal{K} = \{v \in \mathfrak{V} | v(t) \geq 0 \text{ a.e. for a.e. } t \in (0,T)\}$. The domain is given as $\mathcal{Q} = ((0,1)\times(0,1))\times(0,T)$ with $T = 1$. As in Chapter 6 we tested the algorithm for different sets of data with the following parameters.

$$\gamma_0 = 1, \beta_\gamma = 2, \varepsilon_0 = 0.001, \beta_\varepsilon = 0.75, \kappa_0 = 1, \alpha_0 = .001$$

We have plotted the states, controls and multipliers $\lambda_\varepsilon = \lambda_\gamma - \nu\xi - \varepsilon\Delta p$ for different time steps at the last outer loop of the algorithm. In addition we present the norm of the iterates while the outer loop of the algorithm for all values of ε.

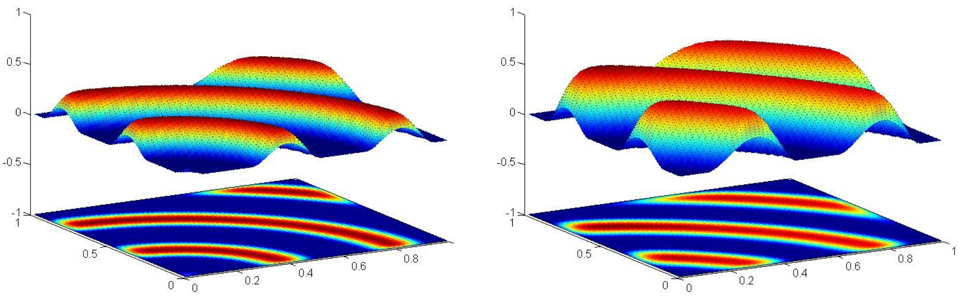
Example 7.1. *In this example, the data are given by a time dependent version of those in Example 6.4. We adopt them to*

$$y^d(x_1,x_2,t) = \cos 15\sqrt{(t+x_1)^2 + (t+x_2)^2}$$

representing a wave moving into the origin. Thus, the subsets of Θ *with* $y^d < 0$ *vary in the domain* Ω *depending on time.*

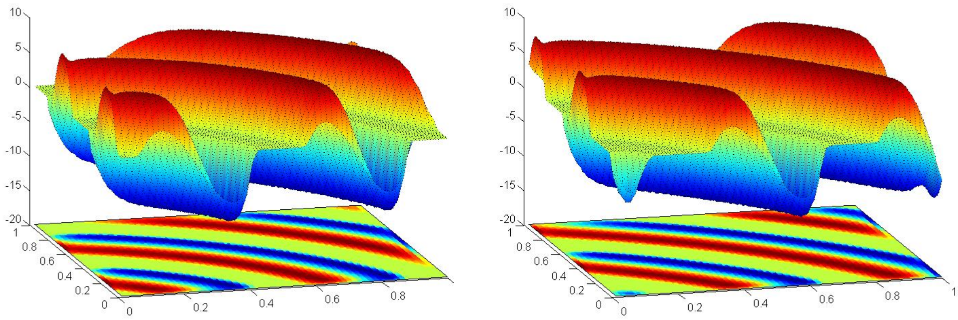
In Figure 7.1, 7.2 and 7.3 we present snapshots of the state y_ε*, the corresponding control* u_ε *and the multiplier* λ_ε *respectively for the final value of* $\varepsilon = 0.001\cdot(0.75)^{33}$ *at time* $3\Delta t$ *and* $28\Delta t$*. Figure 7.4 indicates boundedness of the iterates in the outer loop as assumed.*

Concerning the performance while updating γ*, the algorithm used* 51 *semismooth Newton steps on the coarse and* 26 *on the fine mesh. In the second part most iterations were spend to reconstruct the solution after mesh refinement.*



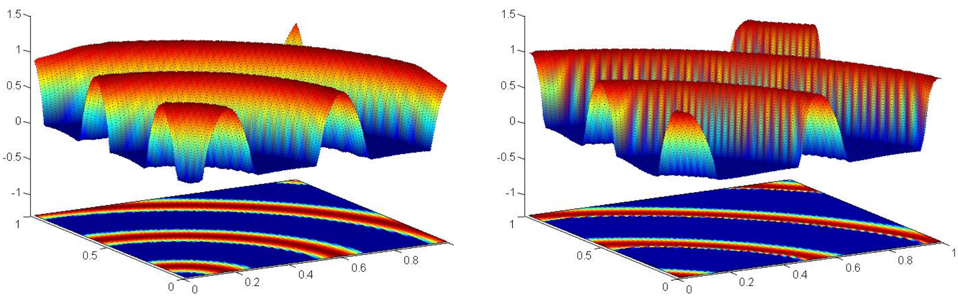
(a) Time Step 3 (b) Time Step 28

Figure 7.1: States of Example 7.1



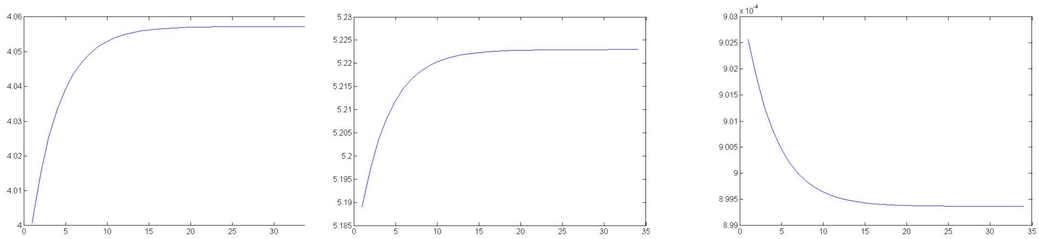
(a) Time Step 3 (b) Time Step 28

Figure 7.2: Control in Example 7.1



(a) Time Step 3 (b) Time Step 28

Figure 7.3: Multiplier of Example 7.1



(a) $\|y_\varepsilon\|$ (b) $|u_\varepsilon|$ (c) $|\xi_\varepsilon|$

Figure 7.4: Norm during ε updating, Example 7.1

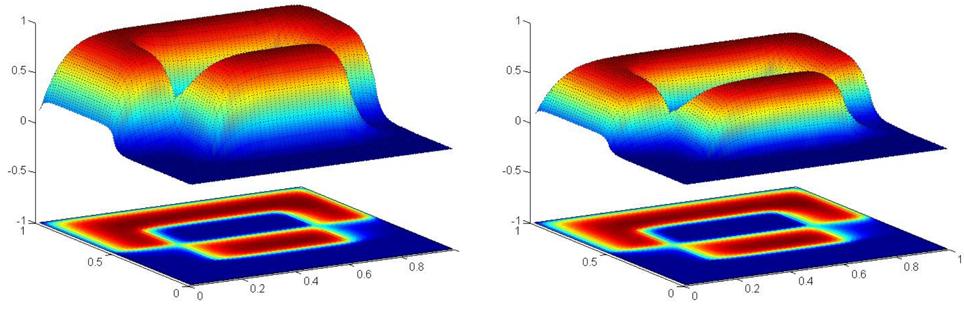
Example 7.2. *Here we consider a time dependent version Example 6.5 with*

$$y^d(x_1,x_2) = \begin{cases} (1+\sin(\pi t))0.5 & \textit{if } x_2 \geq 0.75 \\ (1+\sin(\pi t))0.5 & \textit{if } x_2 \in (0.5,0.75), x_1 \in (0,0.25) \cup (0.75,1) \\ (1+\sin(\pi t))0.5 & \textit{if } x_2 \in (0.25,0.5), x_1 \in (0.25,0.75) \\ 0 & \textit{if } x_2 \in (0.25,0.5), x_1 \in (0,0.25) \cup (0.75,1) \\ -0.25 & \textit{else} \end{cases}$$

This represents a stomping motion increasing the discontinuity of the data and lowering it again.

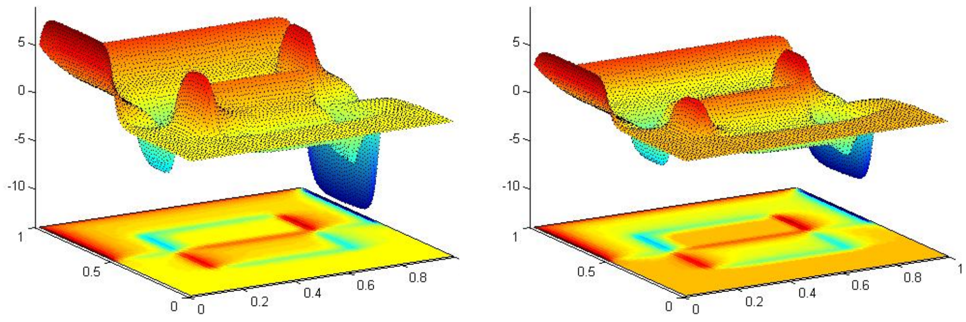
In Figure 7.5, 7.6 and 7.7 we present snapshots of the state y_ε*, the corresponding control* u_ε *and the multiplier* λ_ε *respectively for* $\varepsilon = 0.001 \cdot (0.75)^{33}$ *at time* $30\Delta t$ *and* $60\Delta t$*. Also in this case boundedness of the iterates in the outer loop is indicated by Figure 7.8 in the corresponding discrete norms.*

Before entering the outer loop, the algorithm spent 68 *semismooth Newton iterations on updating* γ *such that only* 16 *steps were performed on the fine mesh.*



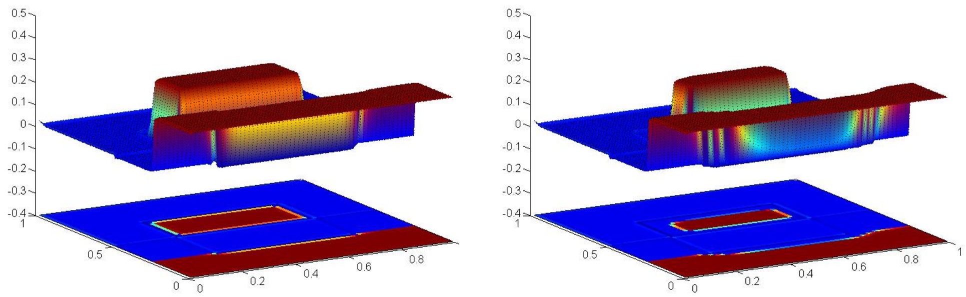
(a) Time Step 30 (b) Time Step 60

Figure 7.5: States of Example 7.2



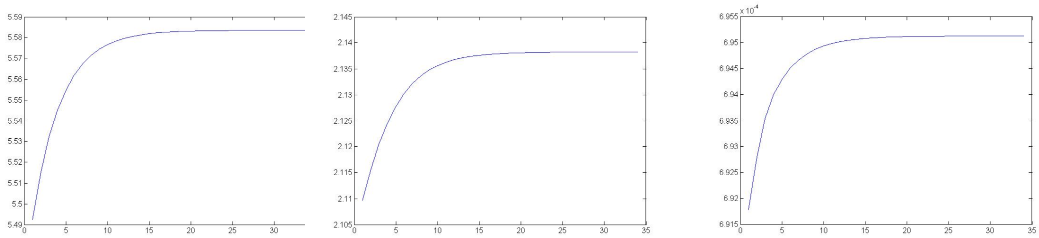
(a) Time Step 30 (b) Time Step 60

Figure 7.6: Control in Example 7.2



(a) Time Step 30 (b) Time Step 60

Figure 7.7: Multiplier of Example 7.2



(a) $\|y_\varepsilon\|$ (b) $|u_\varepsilon|$ (c) $|\xi_\varepsilon|$

Figure 7.8: Norm during ε updating, Example 7.2

Discussion

First we note, that the results from the examples are similar to the outcome of the stationary case in Chapter 6.

Refinement:

The computation time roughly increases ten-fold every time the mesh is refined. By the nature of the algorithm, most of the semismooth Newton steps have to be made on the finest mesh and consequently we restricted ourselves to a discretization of width 2^{-6}. Since the penalization factor is coupled to the mesh we could not increase its value above 16384. Therefore we initialized $\alpha_0 = 0.001$ as small to reconstruct at least a sufficiently small bound on the inner product (y_γ, ξ_γ) which in the end is $\approx 8 \cdot 10^{-6}$. Based on the lack of spatial resolution, the difference to Example 6.4 and 6.5 can be explained. We think that the lack of boundary artifacts as discussed in the preceding chapter is caused by the fact, that γ and κ are not large or small enough respectively and hence they do not occur at this point

Semismooth Newton Method:

The method converged locally with superlinear rate as expected. Similar to the numerical experiments in Chapter 6 we observed in the outer loop either a stable number of iterations for every value of ε or the method failed to converge at all. Figure 7.9 depicts the number of iterations in all steps of the algorithm where ε is decreased in Example 7.2.

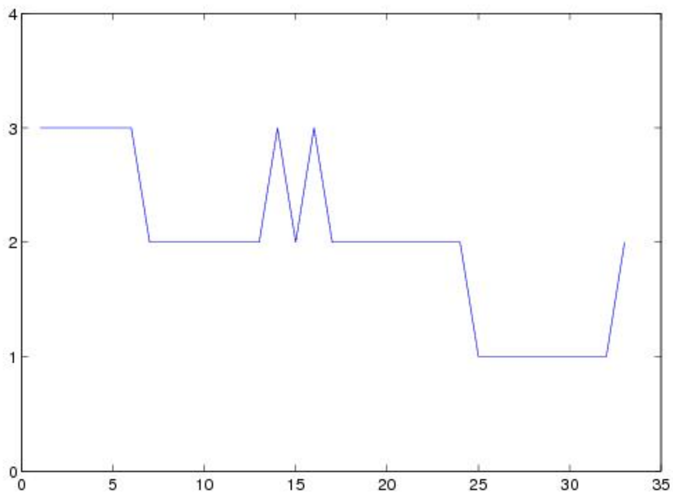
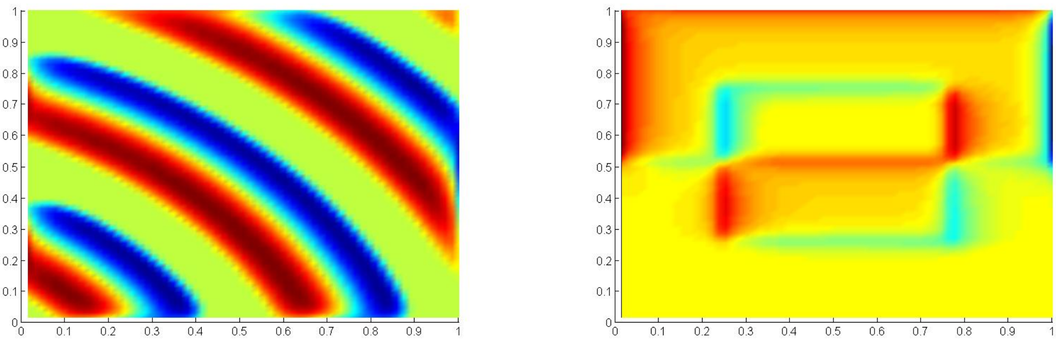


Figure 7.9: Semismooth Newton Iterations per ε decrease

Transport Behavior:

As in Chapter 6 we analyzed the results if they indicate the transport behavior, the first order differential operator induces. In Figure 7.10 we depicted two chosen snapshots of control functions u_ε after the algorithm terminated.



(a) Example 7.1

(b) Example 7.2

Figure 7.10: Top View of Control Functions u_ε in the examples

Note that in the current examples the transport vector has a different sign. Again, red indicates an area where material is allocated while blue represents parts of the domain, where material is removed. Since Example 7.1 is a wave traveling to the origin, the structure of Figure 7.10a is reasonable. In Figure 7.10b the vector **b** can be observed best.

8 Summary and Conclusions

In this thesis we studied problems of optimal control subject to partial differential equations and variational inequalities with first order differential operators of stationary and evolutionary type.
Concerning the optimal control of partial differential equations we introduced a model for open pit mine planning based on continuous functions describing the profiles in Chapter 5. The model is motivated by corresponding theory which was published in [4]. Here, the most challenging constraint on profile functions is a bound ω on the local Lipschitz constant ensuring physical stability in the mine. The model has been reformulated using a change of variables. Due to this reformulation, the optimization variable becomes a time labeling function $\mathcal{T}$, which assigns to each coordinate in the volume its time of excavation. We have proven, that under suitable regularity requirements on $\mathcal{T}$, profile functions of the mine can be globally reconstructed as Lipschitz continuous level functions of $\mathcal{T}$. In the new setting, the constraint ensuring the physical stability becomes a first order partial differential equation which is of Eikonal type and given as

$$\mathcal{T}_z - \frac{1}{\omega}|\nabla\mathcal{T}|_{[\nu]} = u \geq 0.$$

Here the volume is represented by $\Omega \times [0, \bar{z}]$ and $|\cdot|_{[\nu]}$ is the locally smoothed Euclidean norm. It was shown, that any viscosity solution (see [108]) to this equation is physically stable in the sense of [4]. This type of first order partial differential equation can be also interpreted as follows. Considering the profiles as the interface between already excavated and not jet excavated material in the ore body, the evolution of this level set follows an Eikonal equation. However, the derivation of the equation is different from the standard procedure as in [142]. Since we are looking for viscosity solutions to this equation, we discussed a suitable discretization scheme for the considered partial differential equation in Example 4.4 which converges to the correct solution. In addition we established the existence of solutions for the resulting optimization problem of open pit mine planning in the introduced setting. Due to the nature of the first order equation, the solution operator of the underlying equation is not differentiable. Thus we regularized it by an artificial viscosity term with parameter $\varepsilon > 0$. The resulting semilinear parabolic equation can be tackled with available theory (see [98]) and we have proven the existence of solutions to the regularized problems for any viscosity parameter $\varepsilon > 0$. A second constraint of the original problem ensures, that the capacity constraint of the mine is never exceeded by the excavation activities implied by the resulting time labeling function of the optimization problem. We could proof the differentiability of this constraint under the increased regularity of the states for the regularized problems in Proposition 5.2.6. Moreover, as we were able to show directional differentiability of the solution operator for the regularized parabolic partial differential equations, we could establish a first order necessary optimality condition for the regularized problems in Theorem 5.3.2. Finally we proved a mild consistency result in Proposition 5.3.2 showing that locally optimal points of the regularized problems converge to a feasible point of the original problem. We concluded the chapter by presenting numerical results for the ultimate gain pit utilizing a fast sweeping method.

In Chapter 6 we studied problems of optimal control subject to stationary variational inequalities of the first kind with linear first order differential operators. These problems are given as

$$\begin{array}{ll} \min & \frac{1}{2}|y-y^d|^2 + \frac{\tilde{\beta}}{2}\|y\|^2 + \frac{\beta}{2}|u|^2 \\ \text{s.t.} & y \in \mathbf{K}, \;\; \langle \mathbf{b}\cdot\nabla y + b^0 y - f - u, v-y\rangle \geq 0 \quad \forall v \in \mathbf{K} \end{array}$$

with a closed and convex set $\mathbf{K} \subset H_0^1(\Omega)$ and a first order operator satisfying the strong feasibility condition (see Definition 6.1.1).

The increased regularity of the state of the underlying system ($y \in H_0^1(\Omega)$ instead of only $L_B^2(\Omega)$ as introduced in [135]) is crucial for the existence of solutions to the optimization problem since the corresponding proof utilizes the compact embedding of $H_0^1(\Omega)$ into $L^2(\Omega)$. The existence of solutions has been proven in Proposition 6.1.2 for the $L_B^2(\Omega)\cap H_0^1(\Omega)$ case and in Corollary 6.1.1 for the $H_0^1(\Omega)$ case respectively.

We have shown that solutions $y \in H_0^1(\Omega)$ for the underlying variational inequality are unique (see Proposition 6.1.1) and that the variational inequality is equivalent to a complementarity problem (see Lemma 6.1.4). In addition we proved, that any solution of the underlying object admitting the increased regularity $H_0^1(\Omega)$ can be approximated by a sequence of solutions to elliptic variational inequalities where the linear first order operator is regularized by a weighted Laplacian. The proof can be found in Theorem 6.1.2 and the result is essential for showing, that a solution for the original problem exists.

We derived a system similar to $\mathcal{E}$ almost weak stationarity for the original problem although a proof of the condition $\langle\lambda^*, y^*\rangle = 0$ was not possible. This was done in two different function space settings. In Proposition 6.2.3 we established the system for solutions $y \in L_B^2(\Omega)\cap H_0^1(\Omega)$ and obtained the result on the hyperbolic level only, meaning that we did not introduce a regularization of the differential operator. In Theorem 6.2.4 we obtained a slightly stronger system utilizing the procedure to first regularize the linear differential operator with a weighted Laplacian and then to use the penalization and regularization technique on the elliptic level. As a consequence we obtained the desired system by first finding an $\mathcal{E}$ almost C-stationary point for the problems

$$\begin{array}{ll} \min & \frac{1}{2}|y-y^d|^2 + \frac{\tilde{\beta}}{2}\|y\|^2 + \frac{\beta}{2}|u|^2 \\ \text{s.t.} & y \in \mathbf{K}, \;\; \langle -\varepsilon\Delta y + \mathbf{b}\cdot\nabla y + b^0 y - f - u, v-y\rangle \geq 0 \quad \forall v \in \mathbf{K} \end{array}$$

and further discussing a convergence process of such points.

In both cases of the construction of the stationarity system, a certain assumption on the boundedness of the involved functions had to be made. Besides the theoretical results, we presented an algorithm for the second methodology and demonstrated its performance in several numerical examples. Here we could observe, that the assumptions on boundedness, which are crucial for the convergence theory, are met by these examples.

In Section 7 we studied similar problems of optimal control where the differential operator of the underlying variational inequality is of first order in the spatial variables and time dependent. The corresponding problems are given as

$$\begin{array}{lrll} \inf & \frac{1}{2}|y-y^d|^2_{L^2(\mathcal{Q})} + \frac{\tilde{\beta}}{2}\|y\|^2_{L^2(0,T;H_0^1(\Omega))} + \frac{\beta}{2}|u|^2_{L^2(\mathcal{Q})} & & \\ \text{s.t.} & \langle D_t y + \mathbf{b}\cdot\nabla y + b^0 y - f - u, v-y\rangle & \geq 0 & \forall v \in \mathcal{K} \\ & y(0) & = u_0 & \in \mathbf{K}\cap H_0^1(\Omega)\cap H^2(\Omega) \end{array}$$

for a closed and convex set $\mathcal{K}$. In the cases we considered, the set is given as

$$\mathcal{K} = \{\varphi \in L^2(0,T;H_0^1(\Omega))|\varphi(t) \in \mathbf{K} \text{ for almost every } t \in (0,T)\}.$$

We analyzed the existence theory for $\mathbf{K}$ being of obstacle nature as in Chapter 6.
The linear operator is again assumed to carry the strong feasibility condition which is in the time dependent case presented in Definition 7.2.1. We started by studying the underlying variational inequalities and derived the existence of regular solutions $y \in W(0,T)$ by the vanishing viscosity approach under suitable conditions in Lemma 7.2.5. Furthermore we showed, that within $W(0,T)$ solutions are unique. (see Lemma 7.2.3) and that the solution is automatically more regular with respect to time in the sense that it is in fact an element of $\hat{W}(0,T)$ (see Remark 7.2.4).
Finally we showed, that any regular solution $y \in W(0,T)$ of the first order hyperbolic variational inequality for given data $f \in L^2(\mathcal{Q})$ can be approximated in a certain function space by solutions to regularized parabolic problems with the same or even shifted data $f + \varepsilon h$ where ε is the parameter of the regularization and $h \in L^2(\mathcal{Q})$ chosen arbitrarily. The solutions to the regularized variational inequalities

$$\text{find } y \in \mathcal{K} : \langle D_t y - \varepsilon\Delta y + \mathbf{b}\cdot\nabla y + b^0 y - f - u, v - y\rangle \geq 0 \quad \forall v \in \mathcal{K}$$

are naturally elements of $W(0,T)$ and even $\hat{W}(0,T)$ under suitable assumptions (see Theorem 7.2.1 and Lemma 7.2.2). The regularized parabolic variational inequalities are obtained, as in the stationary case, by adding a weighted Laplace operator to the differential operator. The approximation result is presented in Theorem 7.2.3. After investigating the underlying object of the control problem and noting the interesting approximation property we introduced a family of auxiliary problems given as

$$\begin{array}{lrl} \min & \multicolumn{2}{l}{\frac{1}{2}|y - y^d|^2 + \frac{\tilde{\beta}}{2}\|y\|^2_{\mathfrak{Y}} + \frac{\beta}{2}|u|} \\ \text{s.t.} & \langle D_t y - \varepsilon\Delta y + \mathbf{b}\cdot\nabla y + b^0 y - f - u, v - y\rangle & \geq 0 \quad \forall v \in \mathcal{K} \\ & y(0) & = u_0 \in H_0^1(\Omega) \cap H^2(\Omega) \cap \mathbf{K} \end{array}$$

For the approximating and the original problem we proved the existence of solutions. Here, Theorem 7.2.3 played an important role. Moreover, we presented a consistency result for solutions of the approximating problems with the original one where the second major assumption on the problem comes into play. In order to establish the consistency result we had to find a point, independent of ε, which is feasible for the original as well as the auxiliary problems. This was only possible by using a highly regular initial condition and the issue has been addressed in Remark 7.4.1.
For the regularized problems we derived an $\mathcal{E}$ almost W-stationarity system and showed that under suitable assumptions on the boundedness of the functions involved, points satisfying the corresponding systems for $\varepsilon > 0$ converge to a certain kind of stationary point of the original problem similar to $\mathcal{E}$ almost W-stationarity. Again the product condition for state and multiplier could not been established. The proof can be found at Theorem 7.4.3.
We closed with a numerical case study to proof the concept for the case of time dependent variational inequalities with two examples. Again we observed, that the crucial assumption of boundedness of the sequence was satisfied.
Concerning Open Pit Mine planning this thesis developed a useful tool for the handling of slope constraints providing a new way to consider excavation processes. It moved the problem

away from the Integer- and Mixed-Integer Programming formulation towards the area of optimal control subject to partial differential equations and, by the semidiscretization scheme presented in Section 5.5, ordinary differential equations. An interesting open question is the handling of ODE problem which does not fit into the standard theory. Furthermore it is open, whether sensitivity results can be achieved with respect to parameters of the mining operation as for example the discount function.
In the Chapters 6 and 7 we tried to extend the notion of stationarity systems for second order variational inequalities to the case of first order VI's. Here an interesting question is whether the convergence process with respect to γ and ε could be diagonalized. Moreover it is open, if the updating for ε can be controlled similar to path following introduced in Chapter 2. Here the usage of Automatic Differentiation can be of advantage. Finally, preconditioning schemes have to be investigated further

Appendix

1 Lipschitz Continuous Functions

Recall, that a continuous functions $f \in C(\Omega)$, $\Omega \in \mathbb{R}^m$ is called Lipschitz continuous if there exist some constant $c > 0$ such that

$$|f(x) - f(y)| \leq c|x - y|$$

is satisfied. If the constant c depends subsets of Ω, the function is called locally Lipschitz continuous. In addition, we introduce the notation of quasiconvex sets. A set Ω is called quasiconvex if any two points $x, y \in \Omega$ can be joined by a curve γ in Ω such that

$$length(\gamma) = \sup \sum_{i=0}^{N-1} |\gamma(t_{i+1}) - \gamma(t_i)| \leq c|x - y|$$

holds for some constant c which not depends on the points x, y. Here the supremum is taken over all partitions $0 = t_0 < ... < t_i < ... < t_N = 1$ for the curve $\gamma : [0, 1] \to \mathbb{R}^m$. The following result shows an essential equivalence of the Lipschitz continuous functions to a Sobolev space. It can be found in [69].

Theorem 1.1. *For arbitrary domains Ω the space $W^{1,\infty}(\Omega)$ consist of those bounded functions on Ω that are locally Lipschitz continuous. In particular, if Ω is quasiconvex, then $W^{1,\infty}(\Omega)$ consists of all bounded Lipschitz functions on Ω.*

From this result one shows the next result using weak star convergence. (see [69])

Lemma 1.1. *Let f_n be a sequence of Lipschitz continuous functions defined on a quasiconvex domain Ω which converges uniformly. Moreover, let them have a common Lipschitz constant c. Then the limit function is Lipschitz continuous with the same Lipschitz constant c.*

Finally we present a well known result ([52, 69]) concerning differentiability of Lipschitz continuous functions.

Theorem 1.2 (Rademacher)**.** *Let f be a locally Lipschitz continuous functions in Ω. Then f is differentiable almost everywhere.*

2 Embedding Theorems

The part of the Rellich Kondrachov embedding Theorem presented next can be found in [2].

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain and consider $W^{j+m,p}(\Omega)$ such that $mp > n \geq (m-1)p$ holds.*
If Ω has the strong local Lipschitz property, the embedding

$$W^{j+m,p} \to C^{j,\alpha}(\overline{\Omega})$$

with $\alpha \in (0, m - (n/p))$ is compact.

For bounded domains Ω the strong local Lipschitz property reduces to the condition, that for each point $x \in \partial\Omega$ there exist a neighborhood U_x such that $\partial\Omega \cap U_x$ is the graph of a Lipschitz continuous function.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ and a time $T > 0$ be given. Then the following embedding results hold true.*

1) $W(0,T) \to L^2(\mathcal{Q})$ *is compact*

2) $W(0,T) \to C([0,T], L^2(\Omega))$ *is continuous*

3) $W(0,T) \to L^2(0,T; L^{q(n)}(\Omega))$ *is continuous where q depends on n with $q(1) \in [2,\infty]$, $q(2) \in [2,\infty)$ and $q(n) \in [2, 2n/(n-2)]$.*

4) *for $n \le 3$ we have $W(0,T) \to L^3(\mathcal{Q})$*

5) *for $n \ge 4$ we have $W(0,T) \to L^{2+\theta}(\mathcal{Q})$ with $\theta \le n/(n-2) - 1$*

Proof. 1) is a direct application of the Aubin-Lions Lemma and can be found in [144]. 2) can be found in [157]. 3) follows from the embedding Theorems for $H_0^1(\Omega)$ (see [2]). To see 5) use $\theta \le 1$ and consider

$$\int_0^T \int_\Omega |v(x,t)|^{2+\theta} \le \int_0^T \left(\int_\Omega |v(x,t)|^2 \right)^{1/2} \left(\int_\Omega |v(x,t)|^{2+2\theta} \right)^{1/2} = \int_0^T \|v(t)\|_{L^2(\Omega)} \|v(t)\|_{L^{2+2\theta}(\Omega)}^{1+\theta}$$
$$\le \|v\|_{C([0,T];L^2(\Omega))} \int_0^T \|v(t)\|_{L^{2+2\theta}(\Omega)}^{1+\theta} \le c\|v\|_{W(0,T)} \|v\|_{L^{1+\theta}(0,T;L^{2+2\theta}(\Omega))}^{1+\theta}$$
$$\le c\|v\|_{W(0,T)} \|v\|_{L^2(0,T;L^{2+2\theta}(\Omega))}^{1+\theta} \le c\|v\|_{W(0,T)}^{2+\theta}$$

Here we utilized the continuous embedding $L^2(0,T;L^{q(n)}(\Omega)) \to L^{1+\theta}(0,T;L^{q(n)}(\Omega))$ (see, e.g. [157]), 2) and 3). The proof of 4) is analogously to 5) with $\theta = 1$. □

3 Auxilliar Results for PDE's

Recall, that a linear second order differential operator

$$\mathcal{A}y = \sum_{i,j=1}^n a_{i,j}(x) D_{i,j} y + \sum_{i=1}^n a_i(x) D_i y + a(x) y, \quad a_{i,j} = a_{j,i}$$

is called elliptic, if there exist a positive number $\nu > 0$ which actually is the smallest Eigenvalue of the matrix $a_{i,j}$ such that

$$\sum_{i,j=1}^n a_{i,j}(x) \xi_i \xi_j \ge \nu \|\xi\|_{\mathbb{R}^n}$$

holds for all $x \in \Omega$. The following result is known as weak maximum principle and can for example be found in [52, 57].

Theorem 3.1 (Weak Minimum Principle)**.** *Let $\mathcal{A}$ be an elliptic operator on the bounded domain $\Omega \subset \mathbb{R}^n$ with $a(x) \equiv 0$. In addition let*

$$\mathcal{A}y \geq 0 \ \ (\leq 0)$$

hold for all $x \in \Omega$ with $y \in C^2(\Omega) \cap C(\bar{\Omega})$. Then the maximum (minimum) of y is attained on the boundary of Ω, that is

$$\sup_{x\in\Omega} y(x) = \sup_{x\in\partial\Omega} y(x) \ \ (\inf_{x\in\Omega} y(x) = \inf_{x\in\partial\Omega} y(x))$$

4 Analytical Results

The following results are auxiliary analytical results which will be needed in several parts of the thesis. They can be found in [26, 49, 154].

Theorem 4.1. *In separable Banach spaces any closed and convex set is weakly closed.*

Theorem 4.2. *In separable Banach spaces any bounded sequence contains a weakly convergent subsequence.*

Theorem 4.3. *Let E be a Banach space and $\{x_n\}$ a weakly converging sequence with weak limit point x.*
Then $\|x_n\|_E$ is bounded and $\|x\|_E \leq \varliminf_{n\to\infty} \|x_n\|_E$

Theorem 4.4. *Convex and lower semicontinuous implies weak lower semicontinuity.*

The following result is a consequence of the Poincare Friedrichs inequality.

Lemma 4.1. *Let $\Omega \subset \mathbb{R}^n$ be contained in a cube with side length s. Then the following estimates are valid.*

$$\left(\sum_{|\pi|=1} (D^\pi\varphi, D^\pi\varphi)\right)^{1/2} \leq \left(\sum_{|\pi|\leq 1} (D^\pi\varphi, D^\pi\varphi)\right)^{1/2} \leq (1+s)^n \left(\sum_{|\pi|=1} (D^\pi\varphi, D^\pi\varphi)\right)^{1/2}$$

A Banach space $(E, \|\cdot\|_E)$ is called uniformly convex if $\forall\varepsilon > 0$ there exist a $\delta > 0$ such that

$$[x, y \in E, \|x\|_E \leq 1, \|y\|_E \leq 1, \|x - y\| > \varepsilon] \Rightarrow \left\|\frac{1}{2}(x+y)\right\|_E < 1 - \delta$$

is satisfied.

Theorem 4.5. *Assume that E is a uniformly convex Banach space with norm $\|\cdot\|_E$. Let x_n be a sequence in E such that $x_n \rightharpoonup x$ weakly in E and*

$$\limsup_{n\to\infty} \|x_n\|_E \leq \|x\|_E$$

Then $x_n \to x$ strongly in E.

From the parallelogram identity it follows easily, that any Hilbert space is uniformly convex.

Theorem 4.6 (Hölder Inequality). *Let $\Omega \subset \mathbb{R}^n$ and $1 \leq p, q \leq \infty$ satisfying $p^{-1} + q^{-1} = 1$ be given. Consider $f \in L^p(\Omega)$ and $g \in L^q(\Omega)$. We have*

$$fg \in L^1(\Omega) \quad \textit{and} \quad \|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^q(\Omega)}$$

Theorem 4.7 (Hölder Inequality for Bochner spaces). *Let X be a Banach space, $f \in L^p(0,T;X)$ and $g \in L^q(0,T;X)$ with $p^{-1} + q^{-1} = 1$. The we have*

$$\int_0^T |\langle g, f\rangle_{X^*,X}| \leq \left(\int_0^T \|g\|_{X^*}^q \right)^{1/q} + \left(\int_0^T \|f\|_{X^*}^p \right)^{1/p}$$

Theorem 4.8 (Young's Inequality). *For all $x, y \in \mathbb{R}$, $\varepsilon > 0$ and $p, q > 1$ with $1/p + 1/q = 1$ we have*

$$|xy| \leq \varepsilon |x|^p + ((p\varepsilon)^{1-q}/q)|y|^q$$

Theorem 4.9 (Weyl). *Let $\{\varphi_n\}$ be a sequence in $L^p(\Omega)$ and let $\varphi \in L^p(\Omega)$ be given such that $\|\varphi_n - \varphi\|_{L^p(\Omega)} \to 0$ holds.*
Then there exist a subsequence in φ_{n_k} and a function $h \in L^p(\Omega)$ such that

a) $\varphi_{n_k}(x) \to \varphi(x)$ *a.e. on* Ω

b) $|\varphi_{n_k}(x)| \leq h(x)$ *for all k, a.e. on* Ω

Theorem 4.10 (Theorem of Egorov). *Assume, that Ω is a subset of $\mathbb{R}^n$ with finite Lebesgue measure. Let φ_n be a sequence of measurable functions on Ω such that*

$$\varphi_n(x) \to \varphi(x) \textit{ a.e. on } \Omega.$$

Then for all $\tau > 0$ there exist some measurable subset $\Omega_\tau \subset \Omega$ with $|\Omega \backslash \Omega_\tau| \leq \tau$ and $\varphi_n \to \varphi$ uniformly on Ω_τ.

Theorem 4.11 (Fatous Lemma). *Let $\{\varphi_n\}$ be a sequence of functions in $L^1(\Omega)$ that satisfy*

a) *for all n,* $\varphi_n \geq 0$

b) $\sup \int_\Omega \varphi_n \leq \infty$

Then we have $\varphi = \liminf_{n\to\infty} \varphi_n(x) \in L^1(\Omega)$ and

$$\int_\Omega \varphi(x) \leq \liminf_{n\to\infty} \int_\Omega \varphi_n$$

Theorem 4.12 (Variational Lemma). *Let $\Omega \subset \mathbb{R}^n$ be a nonempty open set. Let $v \in L^2(\Omega)$ and suppose*

$$\int_\Omega v\varphi = 0 \textit{ for all } v \in C_c^\infty(\Omega).$$

Then we obtain

$$v(x) = 0 \textit{ almost everywhere on } \Omega.$$

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